

INRADIUS COLLAPSED MANIFOLDS

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ABSTRACT. In this paper, we study collapsed manifolds with boundary, where we assume a lower sectional curvature bound, two sides bounds on the second fundamental forms of boundaries and upper diameter bound. Our main concern is the case when inradii of manifolds converge to zero. This is a typical case of collapsing manifolds with boundary. Actually we show that the inradius collapse occurs when the limit space is a topological closed manifold, for instance. In the general case, we determine the limit spaces of inradius collapsed manifolds as Alexandrov spaces with curvature uniformly bounded below. When the limit space has co-dimension one, we completely determined the topology of inradius collapsed manifold in terms of singular I -bundles. Genral inradius collapse to almost regular spaces are also characterized. In the case of unbounded diameters, we prove that the number of boundary components of inradius collapsed manifolds is at most two. We also discuss the case of non inradius collapse and obtain information on the limit space and a fiber bundle theorem.

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1. INTRODUCTION

In the present paper, we are concerned with collapsing phenomena of Riemannian manifolds with boundary under a lower sectional curvature bound. The study of collapse of closed manifolds has a long history. In the case of two side bounds on sectional curvatures, a deep general theory was established in [7]. Then for the case of lower sectional curvature bound, in [34], [9], [14], the structure of the first Betti numbers and the fundamental groups with their topological rigidity were determined through a fibration theorem. Later on, those results were partly extended to the case of a lower Ricci curvature bound in [5], [6],[8], [15]. Especially the general manifold structure results of lower dimensional collapsed manifolds under a lower sectional curvature bound were established in [29], [36], [30].

In those results, it is crucial to study Alexandrov spaces with curvature bounded below which appear as the Gromov-Hausdorff limit spaces. In particular, Perelman's topological stability theorem has played significant roles. In connection with the study of Alexandrov spaces, the collapsing phenomena of three-dimensional closed Alexandrov spaces with curvature bounded below has been classified in a recent work [18].

For collapsing Riemannian manifolds with boundary, there is a pioneering work by J. Wong [32], [33] on this subject after the investigation in the non-collapsing and bounded curvature case due to [16], [2]. In the study of convergence and collapsing Riemannian manifolds with boundary, it is obvious that the main problem is to control the boundary behavior in a geometric way. It is in [32] that a nice extension procedure over the boundary was first carried out to study collapsed manifolds with boundary under a lower sectional curvature bound. The study of collapse of three-dimensional Alexandrov spaces with boundary is now undergoing in the work [19], where all the details of collapses will be made clear.

In the present paper, partly motivated by [19], we develop and extend results in [33] to a great extent.

Let $\mathcal{M}(n, \kappa, \lambda, d)$ denote the set of all isometry classes of n -dimensional compact Riemannian manifolds M with boundary whose sectional curvature, second fundamental form and diameter satisfy

$$K_M \geq \kappa, |\Pi_{\partial M}| \leq \lambda, \text{diam}(M) \leq d.$$

Every Riemannian manifold in $\mathcal{M}(n, \kappa, \lambda, d)$ can be glued with a warped cylinder along their boundaries in such a way that the resulting space becomes an Alexandrov space with curvature bounded below having C^0 -Riemannian structure and that its boundary is totally geodesic ([32]). Investigating such a cylindrical extension, Wong proved that $\mathcal{M}(n, \kappa, \lambda, d)$ is precompact with respect to the Gromov-Hausdorff distance. He also proved that if $\mathcal{M}(n, \kappa, \lambda, d, v)$ denote the set of all elements $M \in \mathcal{M}(n, \kappa, \lambda, d)$ having volume $\text{vol}(M) \geq v > 0$, then it contains only finitely many homeomorphism types.

Under the situation above, the main problem we are concerned in this paper is as follows:

Problem 1.1. Let M_i be a sequence in $\mathcal{M}(n, \kappa, \lambda, d)$ converging to a length space N with respect to the Gromov-Hausdorff distance.

- (1) Characterize the structure of N ;
- (2) Find geometric and topological relations between M_i and N for large enough i .

The *inradius* of M is defined as the largest radius of metric ball contained in the interior of M :

$$\text{inrad}(M) := \sup_{x \in M} d(x, \partial M).$$

In the present paper, our main concern is the case of $\text{inrad}(M_i)$ converges to zero. We prove in Corollary 3.4 that if $\text{inrad}(M_i)$ converges to zero, then M_i actually dimension collapses in the sense that any limit space N has dimension

$$\dim N \leq n - 1.$$

Therefore in this case, we say that M_i *inradius collapses*. The inradius collapse is a typical case of collapsing of manifolds with boundary. Actually we show that if a sequence M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ converges to a topological closed manifold or a closed Alexandrov space, then M_i inradius collapses (Proposition 7.6).

The main results in this paper are stated as follows. The first one is about the limit spaces.

Theorem 1.2. *Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapses to a length space N with respect to the Gromov-Hausdorff distance. Then N is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, where $c(\kappa, \lambda)$ is a constant depending only on κ and λ .*

It should be noted that M_i are not Alexandrov spaces unless $\Pi_{\partial M_i} \geq 0$, and that the constant $c(\kappa, \lambda)$ really depend on both κ and λ (see Example 3.14 and 3.15).

Let $\mathcal{M}(n, \kappa, \lambda)$ denote the set of all isometry classes of n -dimensional complete Riemannian manifolds M satisfying

$$K_M \geq \kappa, \quad |\Pi_{\partial M}| \leq \lambda.$$

This family is also precompact with respect to the pointed Gromov-Hausdorff convergence. Theorem 1.2 actually holds true for the limit of manifolds in $\mathcal{M}(n, \kappa, \lambda)$ with respect to the pointed Gromov-Hausdorff convergence (see Theorem 6.4).

Next we discuss the topological structure of inradius collapsed manifolds. First consider the case of inradius collapse of codimension one. In this case we define two types of models of the singularities around boundary component of the limit space, *the product or the twisted singular I -fiber bundle* (see Definition 5.1). We can give a complete characterization of codimension one inradius collapsed manifolds as follows.

Theorem 1.3. *Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an $(n - 1)$ -dimensional Alexandrov space N . Then there is a singular I -fiber bundle:*

$$I \rightarrow M_i \xrightarrow{\pi} N.$$

More precisely,

- (1) *If N has no boundary, then M_i is homeomorphic to a product $N \times I$ or a twisted product $N \tilde{\times} I$;*
- (2) *If N has non-empty boundary, each component $\partial_\alpha N$ of ∂N has a neighborhood V such that $\pi^{-1}(V)$ is isomorphic to either the product or the twisted singular I -fiber bundle around $\partial_\alpha N$;*
- (3) *If $\pi^{-1}(V)$ is isomorphic to the product singular I -fiber bundle for some component $\partial_\alpha N$, then M_i is homeomorphic to $D(N) \times [-1, 1]/(x, t) \sim (r(x), -t)$, where r is the canonical reflection of the double $D(N)$.*

Next, we consider inradius collapse to almost regular spaces. We say that an Alexandrov space N is *almost regular* if any point of N has the space of directions whose volume is close to $\text{vol } \mathbb{S}^{\dim N - 1}$.

Theorem 1.4. *Let a sequence M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an Alexandrov space N which is almost regular. Then the topology of M_i can be classified into the following two types:*

- (a) *There exists a locally trivial fiber bundle*

$$F_i \times I \rightarrow M_i \rightarrow N,$$

where F_i is a closed almost nonnegatively curved manifold in a generalized sense as in [34];

- (b) *There exists a locally trivial fiber bundle*

$$\text{Cap}_i \rightarrow M_i \rightarrow N,$$

where Cap_i (resp. ∂Cap_i) is an almost nonnegatively curved manifold with boundary (resp. a closed almost nonnegatively curved connected manifold) in a generalized sense as in [34].

Combined with Proposition 7.6, Theorem 1.4 yields the following.

Corollary 1.5. *Let a sequence M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ converge to a closed manifold of class C^1 with respect to the Gromov-Hausdorff distance. Then we have one of the locally trivial fiber bundles*

$$F_i \times I \rightarrow M_i \rightarrow N, \quad \text{Cap}_i \rightarrow M_i \rightarrow N,$$

where F_i and Cap_i are as in Theorem 1.4.

Corollary 1.5 provides an extension of Theorem 2 in [33], and Theorem 1.4 solves a problem raised in [33], p.297, without assuming the upper sectional curvature bound.

Next we discuss the number of boundary components of inradius collapsed manifolds, where we do not assume the diameter bound.

Theorem 1.6. *There exists a positive number $\epsilon = \epsilon_n(\kappa, \lambda)$ such that if M in $\mathcal{M}(n, \kappa, \lambda)$ satisfies $\text{inrad}(M) < \epsilon$, then*

- (1) *the number k of connected components of ∂M is at most two;*
- (2) *if $k = 2$, then M is diffeomorphic to $W \times [0, 1]$, where W is a component of ∂M .*

Theorem 1.6 (1) was stated in [33], Theorem 5. However it seems to the authors that the argument there is unclear (see Remark 6.1). Theorem 1.6 might be considered as a generalization of the main theorem in [1], where an I -bundle structure was found for an inradius collapsed manifold under the two-sides bound on sectional curvature.

Finally we turn to the case of non inradius collapse. Namely suppose that a sequence M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ converges to a compact length space N while $\text{inrad}(M_i)$ has a positive lower bound. In this case, N is not an Alexandrov space in general. However we can define the notion of *boundary* N_0 of N and the *boundary singular set* \mathcal{S} of N_0 . It is easily seen that the *interior* $N \setminus N_0$ satisfies the local Alexandrov curvature condition $\geq \kappa$. Although the boundary N_0 does not satisfy the local Alexandrov curvature condition at the points of \mathcal{S} , we have

Theorem 1.7. *$N_0 \setminus \mathcal{S}$ equipped with the interior metric is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, where $c(\kappa, \lambda)$ is a constant depending only on κ and λ .*

We call $x \in \mathcal{S}$ of *type 2* if two points $p_i, q_i \in \partial M_i$ with $d^{(\partial M_i)^{\text{int}}}(p_i, q_i)$ uniformly bounded away from 0 converge to x under the convergence $M_i \rightarrow N$.

It is shown in [32] that ∂M for every $M \in \mathcal{M}(n, \kappa, \lambda)$ has a definite lower sectional curvature bound, and therefore the set of all isometry classes of ∂M with its intrinsic metric, $M \in \mathcal{M}(n, \kappa, \lambda, d)$, forms a precompact family, and any limit space is an Alexandrov space.

Theorem 1.8. *Let M_i converges to N while $\text{inrad}(M_i)$ has a positive lower bound. Suppose that*

- (1) *$N \setminus N_0$ is almost regular as an Alexandrov space;*

- (2) $\lim_{GH}(\partial M_i)^{\text{int}}$ is almost regular;
- (3) every point of \mathcal{S} is of type 2.

Then there exists a locally trivial fiber bundle

$$F_i \longrightarrow (M_i) \longrightarrow N^*,$$

where the fibers F_i are closed almost nonnegatively curved manifolds, and N^* is an almost regular Alexandrov space with almost regular boundary having the same Lipschitz homotopy type as N .

Here an Alexandrov space N^* with boundary is called *almost regular with almost regular boundary* if the double $D(N^*)$ of N^* is almost regular. See Theorem 7.13 for more general statement when $N \setminus N_0$ has non-empty boundary as an Alexandrov space.

Note that in Theorem 1.8 the boundary N_0 of N may have singular points of type 2, and therefore N could be very singular at \mathcal{S} . This is why we replace N by a good space N^* . This can be thought of as a kind of resolution of singularities.

The organization and the outline of the proofs are as follows.

In section 2, we first recall basic notions and facts on the Gromov-Hausdorff convergence and Alexandrov spaces with curvature bounded below. Then we focus on Wong's extension procedure of a Riemannian manifold with boundary by gluing a warped cylinder along their boundaries. By [17], the result of the gluing is a $C^{1,\alpha}$ -manifold with C^0 -Riemannian metric, and becomes an Alexandrov spaces with curvature uniformly bounded below. This construction is quite effective and used in an essential way in the present paper.

In section 3, we describe limit spaces of glued Riemannian manifolds with boundary in several aspects. The limit spaces also have gluing structure. In this section we focus on the topological structure of gluing, estimates of multiplicities of gluing, and intrinsic metric structure of the limit space.

In Section 4, we determine the metric structure of limit spaces. First we study the spaces of directions of the limit space at gluing points, and prove that the gluing map preserves the length of curves. This implies that the gluing in the limit space is done metrically in a natural manner, and yields significant structure results (see Theorem 4.19) on the limits including Theorem 1.2.

Those structure results are applied in Section 5 to obtain the fiber structures of inradius collapsed manifolds. Theorems 1.3 and 1.4 are proved there. To prove Theorem 1.3, we need to analyze the singularities of the singular I -fiber bundle in details. To prove Theorem 1.4, we apply an equivariant fibration-capping theorem proved in [36].

To prove Theorem 1.6, we consider the case of unbounded diameters in Section 6. Applying the results in Section 4, we obtain basically three types on local connectedness of the boundary of an inradius collapsed complete manifold, according to the types of the local limit spaces.

After such local observation, Theorem 1.6 follows from a monodromy argument.

In Section 7, we consider the convergence where the inradii have a positive lower bound. In this case, we investigate the "boundary" of the limit space which has the structure of an Alexandrov space with curvature uniformly bounded below outside the "boundary singular point set". In case there are boundary singular points, we classify them into type 1 and 2 singular points. In the case when the boundary singular points consists of only type 2, we obtain a fiber bundle theorem like Theorem 1.8.

2. PRELIMINARIES

In order to make the presented paper more accessible, we fix some basic definition, notations and conventions.

- $\tau(\delta)$ is a function depends on δ such that $\lim_{\delta \rightarrow 0} \tau(\delta) = 0$.
- For topological spaces X and Y , $X \approx Y$ means X is homeomorphic to Y .
- The distance between two points x, y in a metric space is denoted by $d(x, y)$, $|x, y|$ or $|xy|$.
- For a point x and a subset A of a metric space X , $B(x, r) = B^X(x, r)$ and $B(A, r) = B^X(A, r)$ denote open r -balls in X around x and A respectively.
- For a metric space (X, d) , and $r > 0$, the rescaling metric space (X, rd) is denoted by rX .
- The Euclidean cone $K(\Sigma)$ over a metric space (Σ, ρ) is $\Sigma \times [0, \infty)$ equipped with the metric d defined as

$$d((x_1, t_1), (x_2, t_2)) = (t_1^2, t_2^2 - 2t_1t_2 \cos(\min\{\rho(x_1, x_2), \pi\}))^{1/2},$$

for any two points $(x_1, t_1), (x_2, t_2) \in \Sigma \times [0, \infty)$.

- For a subspace M of a metric space $(\tilde{M}, d_{\tilde{M}})$, M^{ext} denotes $(M, d_{\tilde{M}})$, which is called the exterior metric of M .
- The metric d of a connected metric space (X, d) induces a length metric d_{int} of X defined as the infimum of the length of all curves joining given two points. We denote by X^{int} the new metric space (X, d_{int}) .
- The length of a curve γ is denoted by $L(\gamma)$.

2.1. The Gromov-Hausdorff convergence. A (not necessarily continuous) map $f : X \rightarrow Y$ between two metric spaces X and Y is called an ε -approximation if it satisfies

- (1) $|d(x, y) - d(f(x), f(y))| < \varepsilon$, for all $x, y \in Y$,
- (2) $f(X)$ is ε -dense in Y , i.e., $B(f(X), \varepsilon) = Y$.

The Gromov-Hausdorff distance $d_{GH}(X, Y)$ is defined as the infimum of those ε such that there are ε -approximations $f : X \rightarrow Y$ and $g : Y \rightarrow X$.

A map $f : (X, x) \rightarrow (Y, y)$ between two pointed metric spaces is called a *pointed ε -approximation* if it satisfies

- (1) $f(x) = y$,
- (2) $|d(x, y) - d(f(x), f(y))| < \varepsilon$, for all $x, y \in B^X(x, 1/\varepsilon)$,
- (3) $f(B^X(x, 1/\varepsilon))$ is ε -dense in $B^Y(y, 1/\varepsilon)$.

The pointed Gromov-Hausdorff distance $d_{pGH}((X, x), (Y, y))$ is defined as the infimum of those ε such that there are pointed ε -approximations $f : (X, x) \rightarrow (Y, y)$ and $g : (Y, y) \rightarrow (X, x)$.

Consider a pair (X, Λ) of a metric space X and a group Λ of isometries of X . For such pairs (X, Λ) , (Y, Γ) , a triple (f, φ, ψ) of maps $f : X \rightarrow Y$, $\varphi : \Lambda \rightarrow \Gamma$ and $\psi : \Gamma \rightarrow \Lambda$ is called an *equivariant ε -approximation* from (X, Λ) to (Y, Γ) if the following holds

- (1) f is an ε -approximation,
- (2) if $\lambda \in \Lambda$, $x \in X$, then $d(f(\lambda x), (\varphi\lambda)(fx)) < \varepsilon$,
- (3) if $\gamma \in \Gamma$, $y \in Y$, then $d(f(\psi(\gamma)x), \gamma(fx)) < \varepsilon$.

The *equivariant Gromov-Hausdorff distance* $d_{eGH}((X, \Lambda), (Y, \Gamma))$ is defined as the infimum of those ε such that there are ε -approximations from (X, Λ) to (Y, Γ) and from (Y, Γ) to (X, Λ) .

2.2. Alexandrov spaces. Let X be a geodesic metric space, where every two points of X can be joined by a shortest geodesic. For a fixed real number κ and a geodesic triangle Δpqr in X with vertices p , q and r , denote by $\tilde{\Delta}pqr$ a *comparison triangle* in the simply connected model surface M_κ^2 with constant curvature κ . This means that $\tilde{\Delta}pqr$ has the same side lengths as the corresponding ones in Δpqr . Here we suppose that the perimeter of Δpqr is less than $2\pi/\sqrt{\kappa}$ if $\kappa > 0$. The metric space X is called an *Alexandrov space with curvature $\geq \kappa$* , sometimes Alexandrov space for short if we do not emphasis the lower curvature bound, if each point of X has a neighborhood U satisfying the following: For any geodesic triangle in U with vertices p , q and r and for any point x on the segment qr , we have $|px| \geq |\tilde{p}\tilde{x}|$, where \tilde{x} is the point on $\tilde{q}\tilde{r}$ corresponding to x . From now on we assume that an Alexandrov space is always finite dimensional.

For an Alexandrov space X with curvature bounded below by κ , let $\alpha : [0, s_0] \rightarrow X$ and $\beta : [0, t_0] \rightarrow X$ be two geodesics starting from a point x . The *angle* between α and β is defined by $\angle(\alpha, \beta) = \lim_{s, t \rightarrow 0} \tilde{\angle}\alpha(s)x\beta(t)$, where $\tilde{\angle}\alpha(s)x\beta(t)$ denotes the angle of a comparison triangle $\tilde{\Delta}\alpha(s)x\beta(t)$ at the point \tilde{x} . Two geodesics α, β from $x \in X$ is called *equivalent* if $\angle(\alpha, \beta) = 0$. We denoted by $\Sigma'_x(X)$ the set of equivalent classes of geodesics emanating from x . The *space of directions* at x , denoted by $\Sigma_x = \Sigma_x(X)$, is the completion of $\Sigma'_x(X)$ with the angle metric. Let X be n -dimensional. Then Σ_x is an $(n-1)$ -dimensional compact Alexandrov space with curvature ≥ 1 .

A point $x \in X$ is called *regular* if Σ_x is isometric to \mathbb{S}^{n-1} . Otherwise we call x a singular point. We denote by X^{reg} (resp. X^{sing}) the set of all regular points (resp. singular points) of X .

The *tangent cone* at $x \in X$, denoted by $T_x(X)$, is the Euclidean cone $K(\Sigma_x)$ over Σ_x . It is known that $T_x(M) = \lim_{r \rightarrow 0} (\frac{1}{r}M, x)$.

For a closed subset A of X and $p \in A$, the space of directions $\Sigma_p(A)$ of A at p is defined as the set of all $\xi \in \Sigma_p(X)$ which can be written as the limit of directions from p to points p_i in A with $|p, p_i| \rightarrow 0$. For $x, y \in X \setminus A$, consider a comparison triangle on M_{κ}^2 having the side-length $(|A, x|, |x, y|, |y, A|)$ whenever it exists. Then $\angle Axy$ denotes the angle of this comparison triangle at the vertex corresponding to x .

For $x, y, z \in X$, we denote by $\angle xyz$ (resp. $\tilde{\angle} xyz$) the angle between the geodesics yx and yz at x (resp. the geodesics $\tilde{y}\tilde{x}$ and $\tilde{y}\tilde{z}$ at \tilde{x}).

Let X be an n -dimensional Alexandrov space with curvature bounded below by κ . For $\delta > 0$, a system of n pairs of points, $\{a_i, b_i\}_{i=1}^n$ is called an (n, δ) -*strainer* at $x \in X$ if it satisfies

$$\begin{aligned} \tilde{\angle}_{\kappa} a_i x b_i &> \pi - \delta, & \tilde{\angle}_{\kappa} a_i x a_j &> \pi/2 - \delta, \\ \tilde{\angle}_{\kappa} b_i x b_j &> \pi/2 - \delta, & \tilde{\angle}_{\kappa} a_i x b_j &> \pi/2 - \delta, \end{aligned}$$

for every $1 \leq i \neq j \leq n$. If $x \in X$ has a (n, δ) -strainer, then we say x is (n, δ) -strained. In this case, we call x δ -regular. We call X *almost regular* if every point of X is δ_n -regular for some $\delta_n \leq 1/100n$. It is known that a small neighborhood of any almost regular point is almost isometric to an open subset in \mathbb{R}^n .

Inductively on the dimension, the boundary ∂X is defined as the set of points $x \in X$ such that Σ_x has non-empty boundary $\partial \Sigma_x$. We denote by $D(X)$ the double of X , which is also an Alexandrov space with curvature $\geq \kappa$ (see [22]). By definition, $D(X) = X \amalg_{\partial X} X$, where two copies of X are glued along their boundaries.

A boundary point $x \in \partial X$ is called δ -regular if x is δ -regular in $D(X)$. We say that X is *almost regular with almost regular boundary* if every point of X is δ -regular for $\delta < 1/100n$.

In Section 5.1, we need the following result on the dimension of the interior singular point sets.

Theorem 2.1 ([4], cf. [20]).

$$\dim_H(X^{\text{sing}} \cap \text{int} X) \leq n - 2, \quad \dim_H(\partial X)^{\text{sing}} \leq n - 2.$$

Theorem 2.2 ([22], cf. [13]). *If a sequence X_i of n -dimensional compact Alexandrov spaces with curvature $\geq \kappa$ Gromov-Hausdorff converges to an n -dimensional compact Alexandrov space X , then X_i is homeomorphic to X for large enough i .*

A subset E of an Alexandrov space X is called *extremal* ([24]) if every distance function $f = \text{dist}_q$, $q \in M \setminus E$ has the property that if $f|_E$

has a local minimum at $p \in E$, then $df_p(\xi) \leq 0$ for every $\xi \in \Sigma_p(E)$. Extremal subsets posses quite important properties.

Theorem 2.3 ([24]). *Let E be an extremal subset of X .*

- (1) *For every $p \in E$, $\Sigma_p(E)$ is an extremal subset of $\Sigma_p(X)$;*
- (2) *E is totally quasigeodesic in the sense that any nearby two points of E can be joined by a quasigeodesic (see [25]).*
- (3) *E has a topological stratification.*

Theorem 2.3(1), (2) implies the following

Corollary 2.4. *For an extremal subset E of X and $p \in E$, $\dim \Sigma_p(E) \leq \dim E - 1$.*

Suppose that a compact group G acts on X as isometries. Then the quotient space X/G is an Alexandrov space ([4]). Let F denote the set of G -fixed points.

Proposition 2.5 ([24]). *$\pi(F)$ is an extremal subset of X/G , where $\pi : X \rightarrow X/G$ is the projection.*

Boundaries of Alexandrov spaces are typical examples of extremal subsets.

Proposition 2.6 ([36] Prop 5.10). *The boundary ∂X of any finite dimensional Alexandrov space X has a collar neighborhood.*

An n -dimensional Alexandrov space is called *smoothable* if it is a Gromov-Hausdorff limit of n -dimensional closed Riemannian manifolds with a uniform lower sectional curvature bounds.

Theorem 2.7 ([12]). *Let X be a smoothable Alexandrov space. Then for any $p \in X$, every iterated space of directions*

$$\Sigma_{\xi_k}(\Sigma_{\xi_{k-1}}(\cdots(\Sigma_{\xi_1}(\Sigma_p(X))\cdots)),$$

is homeomorphic to a sphere, where

$$\xi_1 \in \Sigma_p(X), \xi_2 \in \Sigma_{\xi_1}(X), \dots, \xi_k \in \Sigma_{\xi_{k-1}}(\cdots(\Sigma_{\xi_1}(\Sigma_p(X))\cdots)).$$

2.3. Manifolds with boundary and gluing. In this section, we consider a Riemannian manifold M with boundary in $\mathcal{M}(n, \kappa, \lambda, d)$. First, we recall some fundamental properties of ∂M , which were derived by Wong[32]. We also recall Wong's cylindrical extension procedure based on Kosovskii's Gluing theorem [17].

Let M be a Riemannian manifold with boundary, and ∂M^α denote a boundary component of ∂M . $(\partial M^\alpha)^{\text{int}}$ means ∂M^α with intrinsic length metric.

The following is a immediate consequence of the Gauss equation.

Proposition 2.8. *For every $M \in \mathcal{M}(n, \kappa, \lambda)$, ∂M has a uniform lower sectional curvature bound: $K_{\partial M} \geq K$, where $K = K(k, \lambda)$.*

Proposition 2.9 ([32]). *Let $M \in \mathcal{M}(n, \kappa, \lambda, d)$.*

- (1) *There exists a constant $D = D(n, k, \lambda, d)$ such that any boundary component ∂M^α has intrinsic diameter bound*

$$\text{diam}((\partial M^\alpha)^{\text{int}}) \leq D;$$

- (2) *∂M has at most J components, where $J = J(n, \kappa, \lambda, d)$;*

It follows from Proposition 2.9 that every boundary component of $M \in \mathcal{M}(n, k, \lambda, d)$ is an Alexandrov space with curvature $\geq K$ and diameter $\leq D$, where $K = K(\kappa, \lambda)$, $D = D(n, \kappa, \lambda, d)$

In general, a Riemannian manifold with boundary is not necessarily an Alexandrov space.

Wong ([32]) carried out a gluing of warped cylinders and M along their boundaries in such a way that the resulting manifold becomes an Alexandrov space having totally geodesic boundary.

This is based on Kosovskii's gluing theorem:

Theorem 2.10 ([17]). *Let M_0 and M_1 be Riemannian manifolds with boundaries Γ_0 and Γ_1 respectively with sectional curvature $K_{M_i} \geq \kappa$ for $i = 0, 1$. Assume that there exists an isometry $\phi : \Gamma_0 \rightarrow \Gamma_1$, and let M denote the space with length metric obtained by gluing M_0 and M_1 along their boundaries via ϕ . Let L_i , $i = 0, 1$, be the second fundamental form of $\Gamma := \Gamma_0 \cong_\phi \Gamma_1 \subset M$ with respect to the normal inward to M_i . Then M is an Alexandrov space with curvature $\geq \kappa$ if and only if the sum $L := L_1 + L_2$ is positive semidefinite.*

Remark 2.11. Actually, for every $\delta > 0$, a smooth Riemannian metric g_δ on M is constructed in [17] in such a way that the sectional curvature of g_δ is greater than $\kappa(\delta)$ with $\lim_{\delta \rightarrow 0} \kappa(\delta) = \kappa$ and that (M, g_δ) Gromov-Hausdorff converges to M as $\delta \rightarrow 0$.

Now let us recall the extension construction in [32].

Suppose M is an n -dimensional compact Riemannian manifold with

$$K_M \geq \kappa, \quad \lambda^- \leq II_{\partial M} \leq \lambda^+.$$

Let $\bar{\lambda} := \min\{0, \lambda^-\}$. Then for arbitrarily $t_0 > 0$ and $0 < \varepsilon_0 < 1$ there exists a monotone non-increasing function $\phi : [0, t_0] \rightarrow \mathbb{R}^+$ satisfying

$$\begin{aligned} \phi''(t) + K\phi(t) &\leq 0, \quad \phi(0) = 1, \quad \phi(t_0) = \varepsilon_0, \\ -\infty < \phi'(0) &\leq \bar{\lambda}, \quad \phi'(t_0) = 0, \end{aligned}$$

for some constant $K = K(\lambda, \varepsilon_0, t_0)$. Now consider the warped product metric on $\partial M \times [0, t_0]$ defined by

$$g(x, t) = dt^2 + \phi^2(t)g_{\partial M}(x)$$

where $g_{\partial M}$ is the Riemannian metric of ∂M induced from that of M . We denote by $\partial M \times_\phi [0, t_0]$ the warped product. It follows from the construction that

$$\bullet \quad II_{\partial M \times \{0\}} \geq |\min\{0, \lambda^-\}|;$$

- $II_{\partial M \times \{t_0\}} \equiv 0$;
- the sectional curvature of $\partial M \times_\phi [0, t_0]$ is greater than a constant $c(\kappa, \lambda^\pm, \varepsilon_0, t_0)$.

Clearly, $\partial M \times \{0\}$ in $\partial M \times_\phi [0, t_0]$ is canonically isometric to ∂M . Thus we can glue M and $\partial M \times_\phi [0, t_0]$ along ∂M and $\partial M \times \{0\}$. The resulting space

$$\tilde{M} := M \amalg_{\partial M} (\partial M \times_\phi [0, t_0])$$

carries the structure of differentiable manifold of class $C^{1,\alpha}$ with C^0 -Riemannian metric ([17]). Obviously M is diffeomorphic to \tilde{M} .

Proposition 2.12 ([32]). *For $M \in \mathcal{M}(n, k, \lambda, d)$, we have*

- (1) \tilde{M} is an Alexandrov space with curvature $\geq \tilde{\kappa}$ and with diameter $\leq \tilde{d}$, where $\tilde{\kappa} = \tilde{\kappa}(\kappa, \lambda)$ and $\tilde{d} = d + 2t_0$;
- (2) the exterior metric M^{ext} is L -bi-Lipschitz homeomorphic to M for the uniform constant $L = 1/\varepsilon_0$.

The notion of warped product also works for metric spaces.

Let X and Y be metric spaces, and $\phi : Y \rightarrow \mathbb{R}_+$ a positive continuous function. Then the warped product $X \times_\phi Y$ is defined as follows (see [31]). For a curve $\gamma = (\sigma, \nu) : [a, b] \rightarrow X \times Y$, the length of γ is defined as

$$L_\phi(\gamma) = \sup_{|\Delta| \rightarrow 0} \sum_{i=1}^k \sqrt{\phi^2(\nu(s_i)) |\sigma(t_{i-1}), \sigma(t_i)|^2 + |\nu(t_{i-1}), \nu(t_i)|^2},$$

where $\Delta : a = t_0 < t_1 < \dots < t_k = b$ and s_i is any element of $[t_{i-1}, t_i]$. The warped product $X \times_\phi Y$ is defined as the topological space $X \times Y$ equipped with the length metric induced from L_ϕ .

Proposition 2.13 ([31], Proposition B.2.6). *Let X_i be a convergent sequence of length spaces. If Y is a compact length space, we have*

$$\lim_{GH}(X_i \times_\phi Y) = (\lim_{GH} X_i) \times_\phi Y.$$

whenever the limits exist.

3. DESCRIPTIONS OF LIMIT SPACES

Under the notations in section 2.3, throughout this section unless otherwise stated, we assume $M_i \in \mathcal{M}(n, k, \lambda)$ Gromov-Hausdorff converges to a compact length space N , where $\text{inrad}(M_i) \rightarrow 0$. Let \tilde{M}_i converge to a compact Alexandrov space Y , and M_i^{ext} converge to a closed subset X of Y under the convergence $\tilde{M}_i \rightarrow Y$.

In this section, we first study the topological structure of Y and show that Y possesses a singular I-bundle structure C/η_0 over N . (Proposition 3.3). We then discuss the intrinsic structure of X and prove that X^{int} is isometric to N (Proposition 3.12).

Here we fix some notations used later on.

- C_i denotes $\partial M_i \times_\phi [0, t_0]$;

- C_{it} denotes the subspace $\partial M_i \times_\phi \{t\}$ in C_i ;
- For $C_i \subset \tilde{M}_i$, C_i^{ext} denotes $(C_i, d_{\tilde{M}_i})$.

3.1. Topological structure of Y . In this subsection, we will prove that Y is homeomorphic to a singular- I -bundle (Proposition 3.3).

Under the notation presented in the begining of this section, in view of Proposition 2.9, passing to a subsequence, we may assume that C_i converges to some compact Alexandrov space C with curvature $\geq K = K(\kappa, \lambda)$. Here C_i is not necessarily connected, and therefore the convergence $C_i \rightarrow C$ should be understood componentwisely. Note that

$$C = C_0 \times_\phi [0, t_0], \quad C_0 = \lim_{i \rightarrow \infty} (\partial M_i)^{\text{int}},$$

where $(\partial M_i)^{\text{int}}$ denotes ∂M_i endowed with length metric induced by its original metric. Usually for simplicity we denote

$$C_0 := C_0 \times \{0\}, \quad C_t := C_0 \times \{t\}.$$

Since the identity map $\iota_i : C_i \rightarrow C_i^{\text{ext}}$ is 1-Lipschitz, we can define a 1-Lipschitz map $\eta : C \rightarrow Y$ in the limits. More precisely, define $\eta : C \rightarrow Y$ by

$$\eta = \lim_{i \rightarrow \infty} g_i \circ \iota_i \circ f_i,$$

where $f_i : C \rightarrow C_i$, $g_i : \tilde{M}_i \rightarrow Y$ are component-wise ε_i -approximations with $\lim \varepsilon_i = 0$. We will prove that η is a quotient map.

From now on, $\eta|_{C_0 \times \{0\}}$ is denoted by η_0 .

The following two lemmas are obvious.

Lemma 3.1. *The map $\eta : C \setminus C_0 \rightarrow Y \setminus X$ is a local isometry.*

Lemma 3.2. *For $(p, t) \in C \setminus C_0$, we have $|\eta(p, t), X| = t$.*

Let C/η_0 denotes the quotient space $C/p \sim q$, where $p \sim q$ if and only if $\eta_0(p) = \eta_0(q)$ for $p, q \in C_0$. By the above results, we obtain a singular- I -bundle structure on Y as follows.

Proposition 3.3. *Y and X are homeomorphic to the quotient spaces C/η_0 and C_0/η_0 respectively.*

Proof. Since C and C_0 are compact, it suffices to prove the following:

- (a) $\eta : C \rightarrow Y$ is surjective,
- (b) $\eta(C_0) = X$,
- (c) $\eta : C \setminus C_0 \rightarrow Y \setminus X$ is injective.

Since C_i^{ext} is ε_i -dense in \tilde{M}_i with $\lim_{i \rightarrow \infty} \varepsilon_i \rightarrow 0$, surjective 1-Lipschitz map $\iota_i : C_i \rightarrow C_i^{\text{ext}}$ converges to the surjective 1-Lipschitz map $\eta : C \rightarrow Y$. This shows (a).

Since $(\partial M_i)^{\text{ext}}$ is ε_i -dense in M_i with $\lim_{i \rightarrow \infty} \varepsilon_i \rightarrow 0$, surjective 1-Lipschitz map $(\partial M_i)^{\text{int}} \rightarrow (\partial M_i)^{\text{ext}}$ converges to the surjective 1-Lipschitz map $\eta_0 : C_0 \rightarrow X$. This shows (b).

Note that C is simply covered by the minimal geodesics from the points of C_{t_0} to C_0 . Since η is injective on C_{t_0} and since η carries those minimal geodesics to minimal geodesics from $\eta(C_{t_0})$ to X , non-branching properties of geodesics in Y implies the injectivity of $\eta : C \setminus C_0 \rightarrow Y \setminus X$. This shows (c). \square

Corollary 3.4. *If M_i inradius collapses to N , then it actually collapses to N . Namely we have $\dim M_i > \dim N$.*

Proof. From Propositions 2.12, 3.3 and Lemma 3.11, we have

$$\begin{aligned} \dim M_i &= \dim \tilde{M}_i \geq \dim Y \\ &\geq \dim X + 1 = \dim N + 1. \end{aligned}$$

\square

Remark 3.5. Wong proved $\dim M_i > \dim N$ in ([33], Lemma 1) under the condition that N is an absolute Poincaré duality space. By Proposition 7.6, we shall show that if N is a closed topological manifold or a closed Alexandrov space, then M_i inradius collapses. Hence Corollary 3.4 give another version of Wong's result.

Recall that Y is homeomorphic to the identification space C/η_0 . We now study the multiplicities of the gluing map η_0 .

Lemma 3.6. $\#\eta_0^{-1}(x) \leq 2$ for every $x \in X$.

Proof. Suppose that $\#\eta_0^{-1}(x) \geq 2$ and take $p_1, p_2 \in \eta_0^{-1}(x)$, and let $y_i := \eta(p_i, t)$, $i = 1, 2$, for some $t > 0$. We show that $|y_i, y_2| = 2t$ or equivalently, $\angle y_1 p y_2 = \pi$ for $t < \phi(t_0)|p_1 p_2|_{C_0}/2$.

Let γ be a minimal geodesic in Y joining y_1 and y_2 . If p_1 and p_2 are contained in distinct connected components of C_0 , γ must meet X , and therefor $|y_i, y_2| = 2t$. Suppose that p_1 and p_2 are contained in the same connected component of C_0 . Let $\tilde{\gamma} = (\sigma, \lambda) : [0, 1] \rightarrow C$ be a shortest path in C such that $\tilde{\gamma}(0) = (p_1, t)$ and $\tilde{\gamma}(1) = (p_2, t)$. Then we have

$$\begin{aligned} L(\gamma) &= L(\tilde{\gamma}) = \int_0^1 \sqrt{\phi^2(\nu(t))|\dot{\sigma}(t)|^2 + |\dot{\nu}(t)|^2} dt. \\ &= \int_0^1 \phi(t_0)|\dot{\sigma}(t)| dt \geq \phi(t_0)|p_1, p_2|_{C_0}. \end{aligned}$$

Thus we have $|(p_1, t), (p_2, t)|_C \geq \phi(t_0)|p_1 p_2|_{C_0}$. If γ does not meet X , the geodesic $\tilde{\gamma} = \eta^{-1} \circ \gamma$ joining (p_1, t) and (p_2, t) in C has the length $L(\gamma) \leq 2t < \phi(t_0)|p_1 p_2|_{C_0}$. This is a contradiction, and therefore γ meets X and $|y_i, y_2| = 2t$ \square

Definition 3.7. In view of Lemma 3.6, we make an identification $N = X^{\text{int}}$ and set for $i = 1, 2$,

$$\begin{aligned} N_i &= X_i := \{x \in X \mid \#\eta_0^{-1}(x) = i\}, \\ C_0^i &:= \{p \in C_0 \mid \eta_0(p) \in X_i\}. \end{aligned}$$

Next we construct a good approximation map $\tilde{M}_i \rightarrow Y$, which helps us to grasp a whole picture on the several convergences.

Let $\psi_i : \partial M_i = C_{i0} \rightarrow C_0$ be an ϵ_i -approximation with $\lim_{i \rightarrow \infty} \epsilon_i = 0$.

Lemma 3.8 ([31]). *The map $\Psi_i : C_i \rightarrow C$ defined by*

$$\Psi_i(p, t) = (\psi_i(p), t)$$

is an ϵ'_i -approximation with $\lim_{i \rightarrow \infty} \epsilon'_i = 0$. Actually, for any approximation map $\Psi'_i : C_i \rightarrow C$ there is a $\psi_i : \partial M_i = C_{i0} \rightarrow C_0$ such that $|\Psi_i(p, t), \Psi'_i(p, t)| < \epsilon'_i$ for $\Psi_i = (\psi_i, \text{id})$.

Proof. This follows from Proposition 2.13. \square

Recall that $\eta : C \setminus C_0 \rightarrow Y \setminus X$ is a locally isometric bijection. In particular for every $y = (p, t_0) \in C_{t_0} \subset Y$, there is a unique minimal geodesic $\gamma_y : [0, t_0] \rightarrow Y$ between X and y such that $\gamma_y(0) \in X$, $\gamma(t_0) = y$. Actually γ_y is defined as $\gamma_y(t) = \eta(p, t)$. Define $g_i^* : C_i^{\text{ext}} \rightarrow Y$ by

$$(3.1) \quad g_i^*(p, t) = \eta \circ \Psi_i \circ \iota_i^{-1}(p, t) = \eta(\psi(p), t).$$

Proposition 3.9. *The map $g_i^* : C_i^{\text{ext}} \rightarrow Y$ defined above provides an ϵ'_i -approximation.*

Let $g_i : C_i^{\text{ext}} \rightarrow Y$ be any ϵ_i -approximation such that $g_i = g_i^*$ on C_{it_0} , namely $g_i(p, t_0) = g_i^*(p, t_0)$.

For the proof of Proposition 3.9, it suffices to show the following.

Lemma 3.10. *$|g_i(p, t), g_i^*(p, t)| < \epsilon'_i$ for all $(p, t) \in C_i^{\text{ext}}$.*

Proof. We have to show that

$$\lim_{i \rightarrow \infty} \sup_{(p, t) \in C_i} |g_i(p, t), g_i^*(p, t)| = 0.$$

Suppose the contrary. Then there are subsequence $\{j\} \subset \{i\}$ and $(p_j, t_j) \in C_j$ such that

$$(3.2) \quad |g_j(p_j, t_j), g_j^*(p_j, t_j)| \geq c > 0,$$

for some constant c independent of j . Passing to a subsequence, we may assume that $(\psi_j(p_j), t_j)$ converges to $(p_\infty, t_\infty) \in C$. Let $\gamma_j(t) = (\psi_j(p_j), t)$, $0 \leq t \leq t_0$, which is a minimal geodesic in C_j^{ext} between ∂M_j and C_{jt_0} . Now $g_j^* \circ \gamma_j(t) = \eta(\psi_j(p_j), t)$ converges to a minimal geodesic $\gamma_\infty(t) = \eta(p_\infty, t)$ realizing the distance between X and $(p_\infty, t_0) \in C_{t_0} \subset Y$.

Since g_j is ϵ_j -approximation, any limit of $g_j \circ \gamma_j$, say $\hat{\gamma}$, must also be minimal geodesic between X and (p_∞, t_0) . From the uniqueness of such geodesic, we have $\gamma_\infty(t) = \hat{\gamma}_\infty(t)$, which contradicts (3.2). \square

3.2. Intrinsic structure of X . In this subsection, we determine the intrinsic structure of X , and prove Proposition 3.12 below, which will be crucial in our start for the classification of Y in terms of N .

Let $X \subset Y$ be the limit of a subsequence of M_i^{ext} under the convergence $\tilde{M}_i \rightarrow Y$. Since $d_{M_i} \geq d_{M_i^{\text{ext}}}$ and M_i is L -bi-Lipschitz homeomorphic to M_i^{ext} for a uniform constant L , we have

The identity $\iota_i : M_i \rightarrow M_i^{\text{ext}}$ is a L -bi-Lipschitz homeomorphism. Therefore passing to a subsequence, we have that

Lemma 3.11. $\iota_i : M_i \rightarrow M_i^{\text{ext}}$ converges to an L -bi-Lipschitz homeomorphism $\iota_\infty : N \rightarrow X$.

Proposition 3.12. X^{int} is isometric to N .

Proof. Recall that L -Lipschitz map $\iota_i : M_i^{\text{ext}} \rightarrow M_i$ converges to a surjective map $h : X \rightarrow N$ satisfying

$$|x, y|_X \leq |h(x), h(y)|_N \leq L|x, y|_X,$$

for every $x, y \in X$. It suffices to show that $|h(x), h(y)|_N \leq |x, y|$. Let $\gamma : [0, \ell] \rightarrow X$ be a minimal geodesic in X joining x to y . For any $\varepsilon > 0$, take a subdivision Δ of γ : $x = x_0 < x_1 < \cdots < x_\alpha < \cdots < x_k = y$ such that denoting by γ_Δ the broken geodesic consisting of minimal geodesic joining $x_{\alpha-1}$ and x_α in Y for $1 \leq \alpha \leq k$, we have

- (1) $|L(\gamma_\Delta) - |x, y|_{X^{\text{int}}}| < \varepsilon$;
- (2) $\max_t |\gamma_\Delta(t), X| < \varepsilon$.

Take $p_\alpha^i \in M_i$ converging to x_α under the convergence $\tilde{M}_i \rightarrow Y$, and denote by γ_Δ^i a broken geodesic consisting of minimal geodesic joining $p_{\alpha-1}^i$ and p_α^i in \tilde{M}_i for $1 \leq \alpha \leq k$. Note that for large enough i

- (1) $|L(\gamma_\Delta) - L(\gamma_\Delta^i)| < \varepsilon$;
- (2) $\max_t |\gamma_\Delta^i(t), M_i| < \varepsilon$.

Let $\sigma_i := \pi_i \circ \gamma_\Delta^i$, where $\pi_i : \tilde{M}_i \rightarrow M_i$ is the canonical projection defined by $\pi_i(p, t) = p$. From the warped product metric construction, we have $L(\gamma_\Delta^i) \geq \phi(\varepsilon)L(\sigma_i)$ for large i . It follows that

$$\begin{aligned} |x, y|_{X^{\text{int}}} &\geq L(\gamma_\Delta) - \varepsilon > L(\gamma_\Delta^i) - 2\varepsilon \\ &\geq \phi(\varepsilon)L(\sigma_i) - 2\varepsilon \\ &\geq \phi(\varepsilon)|p_i, q_i|_{M_i} - 2\varepsilon. \end{aligned}$$

Letting $|\Delta| \rightarrow 0$ and $i \rightarrow \infty$, we conclude that $|x, y|_{X^{\text{int}}} \geq |h(x), h(y)|_N$. This completes the proof. \square

Let $X^{\text{int}} \cup_{\eta_0} C_0 \times_\phi [0, t_0]$ denote the length space obtained by the result of gluing of the two length spaces X^{int} and $C_0 \times_\phi [0, t_0]$ by the map $\eta_0 : C_0 \times 0 \rightarrow X^{\text{int}}$. It is straightforward to see that the canonical map $Y \rightarrow X^{\text{int}} \cup_{\eta_0} C_0 \times_\phi [0, t_0]$ is an isometry. Combined with Proposition 3.12, we have

Proposition 3.13. *Y is isometric to the length space*

$$N \cup_{\eta_0} C_0 \times_{\phi} [0, t_0].$$

3.3. Examples. We exhibit some examples of collapse of manifolds with boundary. All the examples except Example 3.19 are inradius collapses.

Example 3.14. Let $\mathbb{S}^{n-1}(r) := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n (x_i)^2 = r^2\}$. For $\epsilon > 0$, define M_ϵ as the closed domain in \mathbb{R}^n bounded by $\mathbb{S}^{n-1}(r + \epsilon)$ and $\mathbb{S}^{n-1}(r)$. Then $K_{M_\epsilon} \equiv 0$ and $|\Pi_{\partial M_\epsilon}| \leq 1/r$, and M_ϵ inradius collapses to $N := \mathbb{S}^{n-1}(r)$, where the limit space is an Alexandrov space with curvature $\geq r^{-2}$. Note that $N_2 = N$, and that the limit Y of \tilde{M}_ϵ is isometric to the form

$$Y = (\mathbb{S}^{n-1}(r) \amalg \mathbb{S}^{n-1}(r)) \times_{\phi} [0, t_0] / (f(x), 0) \sim (x, 0),$$

where $f : \mathbb{S}^{n-1}(r) \amalg \mathbb{S}^{n-1}(r) \rightarrow \mathbb{S}^{n-1}(r) \amalg \mathbb{S}^{n-1}(r)$ is the canonical involution. Equivalently Y is isometric to the warped product

$$\mathbb{S}^{n-1}(r) \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

This example shows that the lower Alexandrov curvature bound of the limit in Theorem 1.2 really depends on the bound $\lambda \geq |\Pi_{\partial M}|$.

Example 3.15 ([33]). Let $N \subset \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$ be a non-convex domain with smooth boundary, and let M'_ϵ denote the closure of ϵ -neighborhood of N in \mathbb{R}^3 . After a slight smoothing of M'_ϵ , we obtain a flat Riemannian manifold M_ϵ with boundary such that $\Pi_{\partial M_\epsilon} \geq -\lambda$ for some $\lambda > 0$ independent of ϵ . Note that M_ϵ inradius collapses to N , where N has no lower Alexandrov curvature bound.

This example shows that Theorem 1.2 does not hold if one drops the upper bound $\lambda \geq \Pi_{\partial M}$.

Example 3.16. Let $\pi : P \rightarrow N$ be a Riemannian double covering between closed Riemannian manifolds with the deck transformation $\varphi : P \rightarrow P$. Define $\Phi : P \times [-\epsilon, \epsilon] \rightarrow P \times [-\epsilon, \epsilon]$ by

$$\Phi(x, t) = (\varphi(x), -t),$$

and consider $M_\epsilon := P \times [-\epsilon, \epsilon] / \Phi$, which is a twisted I -bundle over N . Note that $M_\epsilon \in \mathcal{M}(n, \kappa, 0, d)$ for some κ and d , and that M_ϵ inradius collapses to N as $\epsilon \rightarrow 0$. In this case, we have $N_2 = N$. Note that the limit Y of \tilde{M}_ϵ is isometric to the form

$$Y = P \times_{\phi} [0, t_0] / (\varphi(x), 0) \sim (x, 0),$$

or equivalently Y is doubly covered by the warped product

$$P \times_{\tilde{\phi}} [-t_0, t_0].$$

Example 3.17. Let N be a convex domain in $\mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^{n+1}$ with smooth boundary. Let M'_ϵ denote the intersection of the boundary of ϵ -neighborhood of N in \mathbb{R}^{n+1} with the upper half space $H_+ = \{(x_1, \dots, x_{n+1}) \mid x_{n+1} \geq 0\}$. After a slight smoothing of M'_ϵ , we obtain a nonnegatively curved Riemannian manifold M_ϵ with totally geodesic boundary. Note that M_ϵ inradius collapses to N as $\epsilon \rightarrow 0$. Note also that $(\partial M_\epsilon)^{\text{int}}$, a smooth approximation of the boundary of ϵ -neighborhood of N in \mathbb{R}^n , converges to the double $D(N)$ of N . It follows that $N_1 = \partial N$ and $N_2 = N \setminus \partial N$, and that the limit Y of \tilde{M}_ϵ is isometric to the form

$$Y = D(N) \times_\phi [0, t_0] / (r(x), 0) \sim (x, 0),$$

where $r : D(N) \rightarrow D(N)$ denotes the canonical reflection of $D(N)$.

Next let us consider more general examples. The following ones come from Example 1.2 in [34], where general examples of collapse of closed manifolds were given.

Example 3.18. Let $\hat{\pi} : M \rightarrow N$ be a fiber bundle over a closed manifold N with fiber F having non-empty boundary and with the structure group G such that

- (1) G is a compact Lie group;
- (2) F has a G -invariant metric g_F of nonnegative curvature which smoothly extends to the double $D(F)$;

Fix a bi-invariant metric b on G and a metric h on N . Let $\pi : P \rightarrow N$ be the principal G -bundle associated with $\hat{\pi} : M \rightarrow N$. Define G -invariant metric g_ϵ on P by

$$g_\epsilon(u, v) = h(d\pi(u), d\pi(v)) + \epsilon^2 b(\omega(u), \omega(v)),$$

where ω is a G -connection on P . Define a metric \tilde{g}_ϵ on $P \times D(F)$ as

$$\tilde{g}_\epsilon = g_\epsilon + \epsilon^2 g_F.$$

For the G -action on $P \times D(F)$ defined by $(p, f) \cdot g = (pg, g^{-1}f)$, \tilde{g}_ϵ is G -invariant and invariant under the action of reflection of $D(F)$. Therefore it induces a metric $g_{D(M), \epsilon}$ on $D(M) = P \times D(F)/G$. Since $g_{D(M), \epsilon}$ is invariant under the action of reflection of $D(M)$, it induces a metric $g_{M, \epsilon}$ on M with totally geodesic boundary such that $(M, g_{M, \epsilon})$ inradius collapse to (N, h) under a lower sectional curvature bound.

Example 3.19. Let M be a compact manifold with boundary, and suppose that a compact Lie group of positive dimension effectively act on M which extends to the action on $D(M)$. Suppose that $D(M)$ has G -invariant and reflection-invariant smooth metric g . As in Example 1.2 of [34], one can construct a metric $g_{D(M), \epsilon}$ on $(D(M))$ which collapses to $(D(M), g_{D(M), \epsilon})/G$ under a lower curvature bound. It follows that the metric $(M, g_{M, \epsilon})$ induced by $g_{D(M), \epsilon}$ also collapses to $(M, g_{M, \epsilon})/G$

under a lower curvature bound. Note that $(M, g_{M,\epsilon})$ has totally geodesic boundary.

4. METRIC STRUCTURE OF LIMIT SPACES

The main purpose of this section is to show that Y and N are actually isometric to C/η_0 and C_0/η_0 respectively. To study how this gluing is made, we first analyze the tangent cones of C , C_0 , Y and X at gluing points, and their relations via the differential $d\eta_0$ of the gluing map η_0 . It turns out that the identification map η_0 preserves length of curves. Finally, we see that N is isometric to a quotient of C_0^{int} by an isometric \mathbb{Z}_2 action. (see Proposition 4.18), which implies Theorems 1.2 and 4.19.

4.1. Spaces of directions and differentials. In this subsection, we study the spaces of directions of C , C_0 , Y and X at the points where the gluing is done, and the relation between them. We also study the differential of the gluing map η_0 at those points.

Let $\tilde{\pi} : C \rightarrow C_0$ and $\pi : Y \rightarrow X$ be the projections. To be precise, $\pi(y) := \eta_0 \circ \tilde{\pi}(\eta_0^{-1}(y))$, which are surjective Lipschitz maps.

Note that $\eta : C \setminus C_0 \rightarrow Y \setminus X$ is a locally isometric bijective map. Therefore $C \setminus C_0$ and $Y \setminus X$ are isometric to each other with respect to their *length* metrics.

For simplicity, we use the same notation

$$C_t := \{q \in C \mid d(C_0, q) = t\}, \quad C_t := \{y \in Y \mid d(X, y) = t\}$$

for every $t \in (0, t_0]$. We also denote by

$$\tilde{\pi}_t : C \rightarrow C_t, \quad \pi_t : Y \setminus X \rightarrow C_t$$

the canonical projections. Recall that

$$X_1 = \{x \in X \mid \#\eta_0^{-1}(x) = 1\}, \quad X_2 = \{x \in X \mid \#\eta_0^{-1}(x) = 2\},$$

$$C_0^i = \{p \in C_0 \mid \eta_0(p) \in X_i\}, \quad i = 1, 2.$$

Lemma 4.1. *For every $p \in C_0$, let $\tilde{\xi}_+ \in \Sigma_p(C)$ be the direction of the minimal geodesic $\tilde{\gamma}_+$ from p to C_{t_0} . Then $\Sigma_p(C)$ is isometric to the half-spherical suspension $\{\tilde{\xi}_+\} * \Sigma_p(C_0)$.*

Proof. Since $C = C_0 \times_\phi [0, t_0]$, obviously we have $T_p(C) = T_p(C_0) \times [0, \infty)$, which implies the conclusion. \square

Lemma 4.2. *For $x \in X_1$, let γ_+ be the shortest geodesic from x to C_{t_0} , and let $\xi_+ \in \Sigma_x(Y)$ be the direction of γ_+ . Then*

(1) *for every $\xi \in \Sigma_x(Y)$, there is a unique $v \in \Sigma_x(X)$ such that*

$$(4.3) \quad \angle(\xi_+, \xi) + \angle(\xi, v) = \angle(\xi_+, v) = \pi/2;$$

(2) *conversely, every $v \in \Sigma_x(X)$ satisfies $\angle(\xi_+, v) = \pi/2$;*

(3) *there exists a unique limit*

$$\lim_{\delta \rightarrow 0} \left(\frac{1}{\delta} X, x \right) = (T_x(X), o_x) = (K(\Sigma_x(X)), o_x).$$

$$\text{under the convergence } \lim_{\delta \rightarrow 0} \left(\frac{1}{\delta} Y, x \right) = (T_x(Y), o_x)$$

Proof. (1) Let γ_+ be the shortest geodesics from x to C_{t_0} , and let $\xi_+ \in \Sigma_x(Y)$ be the directions of γ_+ respectively. First consider an arbitrary minimal geodesic $\gamma : [0, \ell] \rightarrow Y$ starting from x with $\angle(\gamma, \Sigma_x(X)) > 0$. Set $\sigma(t) := \pi(\gamma(t))$, and let $\gamma_t : [0, t_0] \rightarrow Y$ be the minimal geodesic from $\sigma(t)$ to C_{t_0} through $\gamma(t)$. The limit γ_0 of γ_t as $t \rightarrow 0$ coincides with γ_+ . Let $v \in \Sigma_x(X)$ be a direction defined by the curve σ . Let $\tilde{\gamma}, \tilde{\gamma}_+$ (resp. $\tilde{\sigma}$) be geodesics (resp. a curve) in C such that $\eta(\tilde{\gamma}) = \gamma$, $\eta(\tilde{\gamma}_+) = \gamma_+$ (resp. $\eta_0(\tilde{\sigma}) = \sigma$). Since γ is minimal, so is $\tilde{\gamma}$. Note that $\tilde{\sigma}(t) = \tilde{\pi}(\tilde{\gamma}(t))$. Put $p := \tilde{\gamma}(0)$. Let $\tilde{\xi}$ and $\tilde{\xi}_+$ be the directions at p defined by $\tilde{\gamma}$ and $\tilde{\gamma}_+$ respectively. Let \tilde{v} be the directions at p defined by $\tilde{\sigma}$. Note that \tilde{v} is uniquely determined since $\tilde{\sigma}$ is a shortest curve. From Lemma 4.1, we have

$$(4.4) \quad \angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \angle(\tilde{\xi}_+, \tilde{v}) = \pi/2.$$

By the first variation formula, we have $\angle(\xi_+, v) \geq \pi/2$. Now we show (4.3). Consider the rescaling limits,

$$T_x(Y) = \lim_{\delta \rightarrow 0} \left(\frac{1}{\delta} Y, x \right), \quad T_p(C) = \lim_{\delta \rightarrow 0} \left(\frac{1}{\delta} C, p \right).$$

Let σ_∞ and $\tilde{\sigma}_\infty$ be the limits of the Lipschitz curves σ and $\tilde{\sigma}$ under these convergence. It should also be noted that the geodesic γ_δ (resp. $\tilde{\gamma}_\delta$) converges to a geodesic ray $\gamma_{\infty 1}$ (resp. $\tilde{\gamma}_{\infty 1}$) starting from $\sigma_\infty(1)$ (resp. $\tilde{\sigma}_\infty(1)$) and perpendicular to $T_x(X) := \lim_{\delta \rightarrow 0} \left(\frac{1}{\delta} X, x \right) \subset T_x(Y)$ (resp. $T_p(C)$) under those convergences. We set

$$\rho(t) = |C_0, \tilde{\gamma}(t)| = |X, \gamma(t)|.$$

Notice that $\gamma_{\infty 1}$ meet ξ at distance $\rho'(0)$ from $\sigma_\infty(1)$. Thus we have

$$(4.5) \quad \angle(\xi, v) = \text{Arc sin } \rho'(0) = \angle(\tilde{\xi}, \tilde{v}).$$

Since

$$\begin{aligned} |\gamma(\delta), \gamma_+(\rho(\delta))|_{(C_{\rho(\delta)})^{int}} / \delta &\leq |\tilde{\gamma}(\delta), \tilde{\gamma}_+(\rho(\delta))|_{(C_{\rho(\delta)})^{int}} / \delta \\ &= \phi(\rho(\delta))|p, \tilde{\sigma}(\delta)|_{C_0^{int}} / \delta, \end{aligned}$$

as $\delta \rightarrow 0$, we obtain $\sin \angle(\xi_+, \xi) \leq \sin \angle(\tilde{\xi}_+, \tilde{\xi})$, and hence $\angle(\xi_+, \xi) \leq \angle(\tilde{\xi}_+, \tilde{\xi})$. From

$$\begin{aligned} \pi/2 &\leq \angle(\xi_+, v) \leq \angle(\xi_+, \xi) + \angle(\xi, v) \\ &\leq \angle(\tilde{\xi}_+, \tilde{\xi}) + \angle(\tilde{\xi}, \tilde{v}) = \pi/2. \end{aligned}$$

we conclude that (4.3) holds for $\xi \in \Sigma_x(Y) \setminus \Sigma_x(X)$.

Note that (4.3) shows that v is uniquely determined.

(2) For every $v \in \Sigma_x(X)$ take a sequence $x_i \in X$ with $x_i \rightarrow x$, and let $\mu_i : [0, t_i] \rightarrow Y$ denote a minimal geodesic from x to x_i with $v_i := \dot{\mu}_i(0) \rightarrow v$. Let $\lambda_i : [0, t_0] \rightarrow Y$ be a minimal geodesic from x_i to C_{t_0} . We may assume that $\lambda_i(t_0) \rightarrow \gamma_+(t_0)$. Take a point $y_i \in \lambda_i$ such that $|\angle(\xi_+, \xi_i) - \pi/4| < \epsilon_i$ with $\lim \epsilon_i = 0$, where $\xi_i := \uparrow_x^{y_i}$. Let $\gamma_i : [0, s_i] \rightarrow Y$ be a minimal geodesic from x to y_i , and set

$$\sigma_i(t) := \pi(\gamma_i(t)), \quad \tilde{\gamma}_i = \eta^{-1}(\gamma_i), \quad \tilde{\sigma}_i = \tilde{\pi}(\tilde{\gamma}_i).$$

From (1), σ_i defines a direction $\hat{v}_i \in \Sigma_x(X)$ such that

$$\angle(\xi_+, \xi_i) + \angle(\xi_i, \hat{v}_i) = \angle(\xi_+, \hat{v}_i) = \pi/2.$$

Note that $x_i = \sigma_i(s_i)$. Consider the convergence

$$\left(\frac{1}{t_i}Y, x\right) \rightarrow (T_x(Y), o_x), \quad \left(\frac{1}{t_i}C, p\right) \rightarrow (T_p(C), o_p), \quad t_i = |x, x_i|.$$

Then x_i converges to $v \in \Sigma_x(X) \subset T_x(Y)$ under the above convergence. We may assume that ξ_i converge to some $\xi \in \Sigma_x(X) \subset T_x(Y)$.

Passing to a subsequence, we may assume that

- (a) s_i/t_i converges to $s_\infty > 0$;
- (b) $\gamma_i(t_i s)$ and $\sigma_i(t_i s)$ converge to geodesic $\gamma_\infty(s)$ and a Lipschitz curve $\sigma_\infty(s)$ respectively;
- (c) $\tilde{\gamma}_i(t_i s)$ and $\tilde{\sigma}_i(t_i s)$ converge to geodesics $\tilde{\gamma}_\infty(s)$ and $\tilde{\sigma}_\infty(s)$.

We show that σ_∞ is a minimal geodesic. We may also assume that

$$\eta_i = \eta : \left(\frac{1}{t_i}C, p\right) \rightarrow \left(\frac{1}{t_i}Y, x\right)$$

converges to a 1-Lipschitz map

$$\eta_\infty : (T_p(C), o_p) \rightarrow (T_x(Y), o_x),$$

with $\eta_\infty(\tilde{\sigma}_\infty(s)) = \sigma_\infty(s)$. Consider the geodesic triangles

$$\Delta_{o_x} := \Delta_{o_x} \gamma_\infty(s_\infty) \sigma_\infty(s_\infty) \subset T_x(Y),$$

$$\Delta_{o_p} := \Delta_{o_p} \tilde{\gamma}_\infty(s_\infty) \tilde{\sigma}_\infty(s_\infty) \subset T_p(C).$$

Obviously, we obtain

$$\begin{aligned} |o_x, \gamma_\infty(s_\infty)| &= |o_p, \tilde{\gamma}_\infty(s_\infty)|, \\ |\gamma_\infty(s_\infty), \sigma_\infty(s_\infty)| &= |\tilde{\gamma}_\infty(s_\infty), \tilde{\sigma}_\infty(s_\infty)| \end{aligned}$$

Note that $\Sigma_{\gamma_\infty(s_\infty)}(T_x(Y))$ and $\Sigma_{\tilde{\gamma}_\infty(s_\infty)}(T_p(C))$ have the suspension structures and that from construction

$$|\gamma_\infty(s), \sigma_\infty(s)| = |\tilde{\gamma}_\infty(s), \tilde{\sigma}_\infty(s)|.$$

Together with argument in (1), this implies that

$$(4.6) \quad \angle_{o_x} \gamma_\infty(s_\infty) \sigma_\infty(s_\infty) = \angle_{o_p} \tilde{\gamma}_\infty(s_\infty) \tilde{\sigma}_\infty(s_\infty).$$

By the Euclidean cone structure, Δ_{o_x} and Δ_{o_p} spans a flat triangle isometric to ones in \mathbb{R}^2 . From the above equalities, we conclude that $|o_x, \sigma_\infty(s_\infty)| = |o_p, \tilde{\sigma}_\infty(s_\infty)|$. Since $L(\sigma_\infty) \leq L(\tilde{\sigma}_\infty)$, this implies that

σ_∞ is a minimal geodesic, and $\angle(\xi_+, v) = \pi/2$ as required. This also shows that (1) holds true for every $\xi \in \Sigma_x(Y)$.

(3) As observed above, for every $v \in \Sigma_x(X)$ and for every $\epsilon > 0$, one can find a Lipschitz curve σ in X starting from x such that σ determines a well-defined direction $\dot{\sigma}(0) \in \Sigma_x(X)$ satisfying $\angle(v, \dot{\sigma}(0)) < \epsilon$. (3) now follows from a standard argument. \square

Lemma 4.3. *For $x \in X_2$ let γ_\pm be the two shortest geodesics from x to C_{t_0} , and let $\xi_\pm \in \Sigma_x(Y)$ be the directions of γ_\pm respectively. Then we have*

- (1) *for every $\xi \in \Sigma_x(Y)$, there is a unique $v \in \Sigma_x(X)$ such that*
(4.7) $\angle(\{\xi_\pm\}, \xi) + \angle(\xi, v) = \angle(\{\xi_\pm\}, v) = \pi/2;$
(2) *$\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_\pm\} * \Sigma_x(X)$;*
(3) *there exists a unique limit*

$$\lim_{\delta \rightarrow 0} \left(\frac{1}{\delta} X, x \right) = (T_x(X), o_x) = (K(\Sigma_x(X)), o_x).$$

under the convergence $\lim_{\delta \rightarrow 0} (\frac{1}{\delta} Y, x) = (T_x(Y), o_x)$.

Remark 4.4. The suspension structure in Lemma 4.3 (2) also follows from the proof of Lemma 3.6

Proof. (1), (2) In a way similar to (4.3), we have (4.7). Let Σ_\pm denote the union of all minimal geodesics joining ξ_\pm to the elements of $\Sigma_x(X)$. We see that $\Sigma_x(Y)$ is the union of Σ_+ and Σ_- glued along $\Sigma_x(X)$. Therefore $\angle(\xi_+, \xi_-) = \pi$ and $\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_\pm\} * \Sigma_x(X)$.

(3) also follows in a way similar to the proof of Lemma 4.2. \square

For any $p \in C_0$, since η_0 is 1-Lipschitz, $\eta_0 : (\frac{1}{r}C_0, p) \rightarrow (\frac{1}{r}X, x)$ subconverges to a 1-Lipschitz map $(d\eta_0)_p : T_p(C_0) \rightarrow T_x(X)$, which is called a *differential* of η_0 at p .

Proposition 4.5. *For every $p \in C_0$, any differential $d\eta_0 : T_p(C_0) \rightarrow T_x(X)$ satisfies*

$$|d\eta_0(\tilde{v})| = |\tilde{v}|.$$

for every $\tilde{v} \in T_p(C_0)$. In particular, $\eta_0 : C_0 \rightarrow X$ preserves the length of Lipschitz curves in C_0 .

Proof. For every $\tilde{v} \in \Sigma_p(C_0)$, let $\tilde{\xi}$ be the midpoint of a minimal geodesic in $\Sigma_p(C_0)$ joining $\tilde{\xi}_+$ and \tilde{v} . Let $\tilde{\gamma}(t)$ be the geodesic starting from p in the direction $\tilde{\xi}$. Put $\tilde{\sigma}(t) = \tilde{\pi}(\tilde{\gamma}(t))$, $\sigma(t) = \eta_0(\tilde{\sigma}(t))$. Then from (4.5) in the proof of Lemma 4.3, we obtain

$$\tilde{\sigma}'(0) = \frac{\sqrt{2}}{2} \tilde{v}, \quad \sigma'(0) = \frac{\sqrt{2}}{2} v,$$

which implies that $|d\eta_0(\tilde{v})| = |\tilde{v}|$. \square

By Proposition 4.5, $d\eta_0$ provides a surjective 1-Lipschitz map $d\eta_0 : \Sigma_p(C_0) \rightarrow \Sigma_x(X)$.

Remark 4.6. By Lemma 4.3, $x \in X_2$ is a regular point of Y if and only if the tangent cone $T_x(X)$ is isometric to \mathbb{R}^{m-1} , where $m = \dim Y$. From this reason, in that case we call x a *regular point* of X , and set $X^{reg} := X \cap Y^{reg}$. Later we show that every $x \in X_1$ is a *singular point* of X unless $X = X_1$.

Proposition 4.7. *For every $p \in C_0^2$, we have*

- (1) *any differential $d\eta_p$ provides an isometry $d\eta_p : T_p(C) \rightarrow T_x^+(Y)$ which preserves the half suspension structures of both $\Sigma_p(C) = \{\xi_+\} * \Sigma_p(C_0)$ and $\Sigma_x^+(Y) := \{\xi_+\} * \Sigma_x(X)$, where $T_x^+(Y) = T_x(X) \times \mathbb{R}_+$;*
- (2) *$p \in C_0^{reg}$ if and only if $x \in X^{reg}$. In this case, $(d\eta_0)_p : T_p(C_0) \rightarrow T_x(X)$ is a linear isometry.*

Proof. (1) We show that $d\eta_0 : T_p(C_0) \rightarrow T_x(X)$ preserves norm. We use the notation in the proof of Lemma 4.3. Recall that $\eta_0(\tilde{\sigma}(t)) = \sigma(t)$, and $\tilde{v} \in \Sigma_p(C_0)$, $v \in \Sigma_x(X)$ denote the directions determined by $\tilde{\sigma}$, σ respectively. From (4.5), we have

$$|\tilde{\sigma}'(0)| = \cos \angle(\tilde{\xi}, \tilde{v}) = \cos \angle(\xi, v) = |\sigma'(0)|,$$

which implies that $|d\eta_0(\tilde{v})| = |\tilde{v}|$.

For every $\tilde{v}_1, \tilde{v}_2 \in \Sigma_p(C_0)$, put $v_i := d\eta_0(\tilde{v}_i)$. We show that $\angle(\tilde{v}_1, \tilde{v}_2) = \angle(v_1, v_2)$. Let $\tilde{\xi}_i$ (resp. ξ_i) be the midpoint of the geodesic joining $\tilde{\xi}_+$ to \tilde{v}_i (resp. ξ_+ to v_i). Note that $d\eta(\tilde{\xi}_i) = \xi_i$. We may assume that there are geodesics $\tilde{\gamma}_i(t)$ with $\tilde{\gamma}_i'(0) = \tilde{\xi}_i$, and set $\gamma_i(t) := \eta(\tilde{\gamma}_i(t))$. Since $T_x(Y) = T_x(X) \times \mathbb{R}$, any minimal geodesic joining $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(t)$ does not meet X for any small $t > 0$. It follows from the fact that $\eta : C \setminus C_0 \rightarrow Y \setminus X$ is locally isometric that

$$|\tilde{\gamma}_1(t), \tilde{\gamma}_2(t)| = |\gamma_1(t), \gamma_2(t)|,$$

which implies that $\angle(\tilde{\xi}_1, \tilde{\xi}_2) = \angle(\xi_1, \xi_2)$. From the suspension structures, we conclude that $\angle(\tilde{v}_1, \tilde{v}_2) = \angle(v_1, v_2)$.

(2) is an immediate consequence of (1). \square

4.2. Gluing maps. Using the results of the last subsection, we study the metric properties of the gluing map.

From Lemma 3.6, we can define a map $f : C_0 \rightarrow C_0$ as follows: For an arbitrary point $p \in C_0$, let $f(p) := q$ if $\{p, q\} = \eta_0^{-1}(\eta_0(p))$, where q may be equal to p if $\eta_0(p) \in X_1$. Note that f is an involutive map, i.e., $f^2 = id$. Moreover

Lemma 4.8. *$f : C_0 \rightarrow C_0$ is a homeomorphism.*

Proof. Since f is involutive, it suffices to prove that f is continuous. Let a sequence p_i converges to a point p in C_0 . Passing to a subsequence,

we assume that $q_i := f(p_i)$ converges to a point q in C_0 . We have to prove that $f(p) = q$. We observe that

$$(4.8) \quad \eta_0(p) = \eta_0(q).$$

First consider the case $p \in C_0^1$, or equivalently $f(p) = q$. From (4.8), we certainly obtain $p = q = f(p)$.

Next consider the case $p \in C_0^2$. Suppose $f(p) \neq q$. Since $f(p) \neq p$, we then have $p = q$. Let $\tilde{\gamma}_i$, $\tilde{\gamma}_+$, $\tilde{\gamma}_-$ and $\tilde{\gamma}'_i$ be minimal geodesics to C_{t_0} starting from p_i , p , $f(p)$ and q_i respectively. Put $\gamma_i := \eta(\tilde{\gamma}_i)$, $\gamma_+ := \eta(\tilde{\gamma}_+)$, $\gamma_- := \eta(\tilde{\gamma}_-)$ and $\gamma'_i := \eta(\tilde{\gamma}'_i)$. Set $x_i := \eta_0(p_i) = \eta_0(q_i)$, $x := \eta_0(p) = \eta_0(q)$. By Lemma 4.3, we have

$$(4.9) \quad \tilde{Z}\gamma_+(s_0)x\gamma_-(s_0) > \pi - \tau(s_0)$$

By Lemma 5.6 in [4],

$$(4.10) \quad |\angle xx_i\gamma_\pm(s_0) - \tilde{Z}xx_i\gamma_\pm(s_0)| < \tau(s_0) + o_i$$

$$(4.11) \quad |\angle xx_i\gamma_\pm(s_0) - \pi/2| < \tau(s_0) + o_i,$$

where $\lim_{i \rightarrow \infty} o_i = 0$. Let $\sigma_i^\pm : [0, \ell_i^\pm] \rightarrow Y$ be minimal geodesic joining x_i to $\gamma_\pm(s_0)$.

Suppose that $p_i \in C_0^2$. Since $\Sigma_{x_i}(Y)$ is isometric to the spherical suspension $\{\dot{\gamma}_i(0), \dot{\gamma}'_i(0)\} * \Sigma_x(X)$, in view of (4.11), we may assume that $\angle(\dot{\gamma}'_i(0), \dot{\sigma}_i^-(0)) < \tau(s_0) + o_i$. This implies that $|\gamma'_i(s_0), \gamma_-(s_0)| < (\tau(s_0) + o_i)s_0$. Since $|\gamma'_i(s_0), \gamma_+(s_0)| < o_i$, this yields a contradiction to (4.9).

Finally suppose that $p_i \in C_0^1$. Let γ_t^\pm be the minimal geodesic from $\pi(\sigma_i^\pm(t))$ to C_{t_0} . As $t \rightarrow 0$, γ_t^\pm converges to minimal geodesics γ_0^\pm from x_i to C_{t_0} . Since $|\gamma_0^\pm(s_0), \gamma_\pm(s_0)| < o_i$, it follows that $|\gamma_0^+(s_0), \gamma_0^-(s_0)| > 2s_0 - o_i$. In particular we have $\gamma_0^+ \neq \gamma_0^-$, which contradicts to $p_i \in C_0^1$. \square

Corollary 4.9. $\eta_0|_{C_0^2} : C_0^2 \rightarrow X_2$ is a double covering space and X_2 is open in X .

Proof. For $x \in X_2$ set $\eta_0^{-1}(x) = \{p_1, p_2\}$, and take a neighborhood D_1 of p_1 in C_0 such that $D_1 \cap f(D_1)$ is empty. We set $D_2 = f(D_1)$. We show that $E := \eta_0(D_i)$ is open in X . Suppose that E is not open, and take $y \in E$ for which there are $y_i \in X \setminus E$ converging to y . Choose any $q_i \in \eta_0^{-1}(y_i)$. Passing to a subsequence, we may assume that q_i converges to a point q . It turns out that $\eta_0^{-1}(y)$ contains at least three points q_1, q_2 and q , where $q_i \in D_i$, $q \notin D_1 \cup D_2$. Since this is a contradiction, E is open. Now it is immediate that each restriction $\eta_0|_{D_i} : D_i \rightarrow E$ is a homeomorphism. \square

Corollary 4.10. Y and X are homeomorphic to the quotient spaces $C_0 \times_\phi [0, t_0]/f$ and C_0/f respectively, where $(x, 0)$ and $(f(x), 0)$ are identified for every $x \in C_0$.

Corollary 4.11. *If the inradius of $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ converges to zero, then the number of components of ∂M_i is at most two for large enough i .*

Proof. Since f is an involutive homeomorphism, f gives a transposition of two components of C_0 . The conclusion is immediate from the connectedness of X . \square

Remark 4.12. In Theorem 1.6, we remove the diameter bound to get the diameter free result.

Lemma 4.13. $\eta_0|_{C_0^2} : (C_0^2)^{\text{int}} \rightarrow X_2^{\text{int}}$ is a local isometry.

Proof. Since $\eta_0|_{C_0^2} : C_0^2 \rightarrow X_2$ is a covering by Corollary 4.9, we can find relatively compact open subsets D and E of C_0^2 and X^2 respectively such that $\eta_0 : D \rightarrow E$ is a homeomorphism. We must show that $\eta_0 : D \rightarrow E$ is an isometry with respect to the interior distances of C_0 and X respectively. Since η_0 is 1-Lipschitz, it suffices to show that $g := \eta_0^{-1} : E \rightarrow D$ is 1-Lipschitz. We may assume that D is small enough so as to satisfy that for every $x, y \in E$, there is a minimal geodesic $\gamma : [0, 1] \rightarrow X_2$ joining x to y . We do not know if $g \circ \gamma$ is a Lipschitz curve yet. However by Proposition 4.5, $g \circ \gamma$ has the speed $v_{g \circ \gamma}(t)$ (see [3])

$$v_{g \circ \gamma}(t) = \lim_{\epsilon \rightarrow 0} \frac{|g \circ \gamma(t), g \circ \gamma(t + \epsilon)|}{|\epsilon|},$$

which is equal to the speed $v_\gamma(t)$ of γ , and therefore

$$|x, y| = L(\gamma) = \int_0^1 v_{g \circ \gamma}(t) dt = L(g \circ \gamma) \geq |g(x), g(y)|.$$

This completes the proof. \square

Recall that $C_0^i := \{p \in C_0 \mid \eta_0(p) \in X_i\}$, $i = 1, 2$.

Lemma 4.14. *If X_1 has non-empty interior in X , then $X = X_1$ and $\eta_0 : (C_0)^{\text{int}} \rightarrow X^{\text{int}}$ is an isometry.*

Proof. If the interior U of X_1 is non-empty, then $V := \eta_0^{-1}(U) \subset C_0^1$ is open in C_0 . From the non-branching property of geodesics in Alexandrov spaces, we have $V = C_0$ and $X = X_1$. An argument similar to the proof of Lemma 4.13 shows that $\eta_0 : (C_0)^{\text{int}} \rightarrow X^{\text{int}}$ is an isometry. \square

Proposition 4.15. $f : (C_0)^{\text{int}} \rightarrow (C_0)^{\text{int}}$ is an isometry.

Proof. For $x \in X_2$ with $\eta_0^{-1}(x) = \{p_1, p_2\}$, by lemma 4.13, we can take disjoint open sets $p_i \in D_i$, $i = 1, 2$, and E such that $\eta_0^i = \eta_0|_{D_i} : D_i \rightarrow E$ are isometry. Thus $f|_{D_1} = (\eta_0^2)^{-1} \circ \eta_0^1 : D_1 \rightarrow D_2$ is an isometry with respect to the interior distances. Note that f is identity on C_0^1 , and by Lemma 4.13, $f : (C_0^2)^{\text{int}} \rightarrow (C_0^2)^{\text{int}}$ is a locally isometry. For every $p_1, p_2 \in C_0$ we show that $|f(p_1), f(p_2)| = |p_1, p_2|$. This is obvious

if $p_1, p_2 \in C_0^1$. Let $\gamma : [0, 1] \rightarrow C_0$ be a minimal geodesic joining p_1 to p_2 . If $p_1, p_2 \in C_0^2$, applying Lemma 4.13, we may assume that γ meets C_0^1 . Let $t_0 \in (0, 1)$ be the smallest parameter with $\gamma(t_0) \in C_0^1$. By Lemma 4.13, we have $|f(p_1), f(\gamma(t_0))| = |p_1, \gamma(t_0)|$. Therefore the non-branching property of geodesics in Alexandrov space implies that $\gamma \cap C_0^1$ consists of only the single point $\gamma(t_0)$, and therefore we also have $|f(p_2), f(\gamma(t_0))| = |p_2, \gamma(t_0)|$. It follows that

$$\begin{aligned} |f(p_1), f(p_2)| &\leq |f(p_1), f(\gamma(t_0))| + |f(\gamma(t_0)), f(p_2)| \\ &\leq |p_1, \gamma(t_0)| + |\gamma(t_0), p_2| = |p_1, p_2|. \end{aligned}$$

Repeating this, we also have $|p_1, p_2| \leq |f(p_1), f(p_2)|$, and $|f(p_1), f(p_2)| = |p_1, p_2|$. The case of $p_1 \in C_0^1$ and $p_2 \in C_0^2$ is similar, and hence is omitted. This completes the proof. \square

4.3. Structure theorems. In this subsection, making use of the results on gluing maps in the last subsection, we obtain structure results for limit spaces.

We begin with

Lemma 4.16. *X_2 is convex in X .*

Proof. Suppose this is not the case. Then we have a minimal geodesic $\gamma : [0, 1] \rightarrow X$ joining points $x, y \in X_2$ such that γ is not entirely contained in X_2 . Let t_1 be the first parameter with $\gamma(t_1) \in X_1$. Set $z := \gamma(t_1)$. By Lemma 4.13, for any $p \in \eta_0^{-1}(x)$, there exists a unique geodesic $\tilde{\gamma} : [0, t_1] \rightarrow C_0$ such that $\tilde{\gamma}(0) = p$ and $\eta_0 \circ \tilde{\gamma}(t) = \gamma(t)$, for every $t \in [0, t_1]$. Put $\tilde{z} := \tilde{\gamma}(t_1) \in C_0^1$, and take $\tilde{v} \in \Sigma_{\tilde{z}}(C_0)$ such that $(d\eta_0)_{\tilde{z}}(\tilde{v}) = \frac{d}{dt}\gamma(t_0) \in \Sigma_z(X)$. Let $\tilde{\gamma}_1 : [0, t_1] \rightarrow C_0$ and $\gamma_1 : [0, t_1] \rightarrow X$ be the reversed geodesic to $\tilde{\gamma}$ and $\gamma_{[0, t_1]}$: $\tilde{\gamma}_1(t) = \tilde{\gamma}(t_0 - t)$, $\gamma_1(t) = \gamma(t_1 - t)$, and set $\tilde{\gamma}_2(t) := f(\tilde{\gamma}_1(t))$. Since $(d\eta_0)_{\tilde{z}}$ preserves norm and is 1-Lipschitz, we have

$$\angle(\tilde{v}, \tilde{\gamma}'_i(0)) \geq \angle\left(\frac{d}{dt}\gamma(t_1), \frac{d}{dt}\gamma_1(0)\right) = \pi,$$

for $i = 1, 2$. Since $\tilde{\gamma}'_1(0) \neq \tilde{\gamma}'_2(0)$, this is impossible in the Alexandrov space C_0 . \square

Lemma 4.17. *For every $x, y \in X$, let $\gamma : [0, 1] \rightarrow X$ be a minimal geodesic joining x to y , and let $p \in C_0$ be such that $\eta_0(p) = x$. Then there exists a unique minimal geodesic $\tilde{\gamma} : [0, 1] \rightarrow C_0$ starting from p such that $\eta_0 \circ \tilde{\gamma} = \gamma$.*

In particular, if X_1 is not empty, then C_0 is connected.

Proof. From Lemmas 4.16 and the discussion there using non-branching property of geodesics in Alexandrov spaces, we have only the following possibilities:

- (1) γ is included in X_1 or X_2 ;

- (2) only one end point of γ is contained in X_1 and the other part of γ is included in X_2 .

The conclusion follows immediately from Lemmas 4.13 and 4.14. \square

Proposition 4.18. X^{int} is isometric to C_0^{int}/f .

Proof. In the case of $X = X_1$ or $X = X_2$, the conclusion follows from Lemma 4.14 or Proposition 4.13 respectively. Next assume that both X_1 and X_2 are non-empty. We set $Z := C_0^{\text{int}}/f$, which is an Alexandrov space, and decompose Z as

$$Z = Z_1 \cup Z_2, \quad Z_i := C_0^i/f, \quad i = 1, 2.$$

For every $[p] \in Z_1$, $\Sigma_{[p]}(Z)$ is isometric to $\Sigma_p(C_0)/f_*$, where $f_* : \Sigma_p(C_0) \rightarrow \Sigma_p(C_0)$ is an isometry induced by f . Since X_1 is a proper subset of X , f_* defines a non-trivial isometric \mathbb{Z}_2 -action on $\Sigma_p(C_0)$. Thus $[p]$ is a single point of Z : $[p] \in Z^{\text{sing}}$, and therefore $Z_1 \subset Z^{\text{sing}}$. Thus $Z^{\text{reg}} \subset Z_2$. Now by Proposition 4.7, there exists an isometry $F_0 : Z_2 \rightarrow X_2^{\text{int}}$. Since Z^{reg} is convex in Z (see [27]), F_0 defines a 1-Lipschitz map $F_1 : (Z^{\text{reg}})^{\text{ext}} \rightarrow X$ which extends to a 1-Lipschitz map $F : Z \rightarrow X$, where $(Z^{\text{reg}})^{\text{ext}}$ denotes the exterior metric of Z^{reg} .

Conversely since X_2 is convex in X by Lemma 4.16, F_0^{-1} defines a 1-Lipschitz map $G_1 : (X_2)^{\text{ext}} \rightarrow Z_2$ which extends to a 1-Lipschitz map $G : X \rightarrow Z$ satisfying $G \circ F = 1_Z$. Therefore X must be isometric to Z . \square

Proof of Theorem 1.2. By Proposition 4.15, $f : C_0^{\text{int}} \rightarrow C_0^{\text{int}}$ is an involutive isometry. By Propositions 3.12 and 4.18, N is isometric to C_0^{int}/f . Since C_0^{int} is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, so is N . \square

Theorem 4.19. Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to a compact length space N . Let \tilde{M}_i Gromov-Hausdorff converge to Y , and M_i^{ext} converge to $X \subset Y$ under the convergence $\tilde{M}_i \rightarrow Y$. Then

- (1) X^{int} is isometric to N ;
- (2) Y is isometric to $C_0^{\text{int}} \times_{\phi} [0, t_0]/(f(x), 0) \sim (x, 0)$, or equivalently, isometric to the following quotient by an isometric involution $\tilde{f} = (f, -\text{id})$.

$$C_0^{\text{int}} \times_{\tilde{\phi}} [-t_0, t_0]/\tilde{f},$$

where $\tilde{\phi}(t) = \phi(|t|)$.

In particular, Y is a singular I -bundle over N , where singular fibers occur exactly on X_1 unless $X = X_1$.

Compare Examples 3.14, 3.16 and 3.17.

Proof of Theorem 4.19. (1) is just Proposition 3.12. (2) follows immediately from Propositions 3.13 and 4.18. \square

Proposition 4.20. *If $x \in X_1$, then $\Sigma_x(X)$ is isometric to the quotient space $\Sigma_p(C_0)/f_*$, and $\Sigma_x(Y)$ is isometric to the quotient space $\Sigma_p(C)/f_*$, where $f_* : \Sigma_p(C_0) \rightarrow \Sigma_p(C)$ is an isometry induced by f .*

Proof. Take an f -invariant neighborhood U_p of p in C_0 , where $\eta_0(p) = x$. It is easy to check that $V_x := \eta_0(U_p)$ is a neighborhood of x isometric to U_p/f . The conclusion of (2) follows immediately. \square

Corollary 4.21. *Let $\dim N = m$. Suppose that both X_1 and X_2 is non-empty. Then every element $x \in X_1$ satisfies that*

$$\text{vol } \Sigma_x(X) \leq \frac{1}{2} \text{vol } \mathbb{S}^{m-1}.$$

Proof. For $x \in X_1$, take $p \in C_0$ with $\eta_0(p) = x$. Note that C_0 is connected by Lemma 4.17. If $f_* : \Sigma_p(C_0) \rightarrow \Sigma_p(C)$ is the identity, then the non-branching property of geodesics in Alexandrov spaces implies that f is the identity on C_0 . Therefore f_* must be non-trivial on $\Sigma_p(C_0)$. The conclusion follows since

$$\text{vol } \Sigma_x(X) = (1/2) \text{vol } \Sigma_p(C_0) \leq (1/2) \text{vol } \mathbb{S}^{m-1}.$$

\square

By Corollary 4.21, if every $x \in X$ satisfies that

$$\text{vol } \Sigma_x(X) > (1/2) \text{vol } \mathbb{S}^{m-1},$$

then $X = X_1$ or $X = X_2$.

Next let us consider such a case. If $X = X_1$, then by Lemma 4.14, η_0 is an isometry. If $X = X_2$, then by Lemma 4.13, η_0 is a locally isometric double covering. Therefore it is straightforward to see the following.

Corollary 4.22. *If $X = X_1$ or X_2 , then Y can be classified by N as follows.*

- (1) *if $X = X_1$, then Y is isometric to $N \times_\phi [0, t_0]$.*
- (2) *if $X = X_2$, then either Y is isometric to the gluing*

$$N \times_{\tilde{\phi}} [-t_0, t_0],$$

with length metric, or else, Y is a nontrivial I -bundle over N , and is doubly covered by

$$C_0^{\text{int}} \times_{\tilde{\phi}} [-t_0, t_0],$$

where $\tilde{\phi}(t) = \phi(|t|)$.

Compare Examples 3.14 and 3.16.

From now, we write for simplicity as $C_0 := C_0^{\text{int}}$.

5. INRADIUS COLLAPSED MANIFOLDS

In this section, we investigate the structure of inradius collapsed manifolds M_i applying the structure results for limit spaces in Section 4. First we study the case of inradius collapse of codimension one to determine the manifold structure. To carry out this, some additional considerations on the limit spaces are needed to determine the singularities of singular I -fibered spaces. In the second part of this section, we study inradius collapse to almost regular spaces.

5.1. Inradius collapse of codimension one. We consider $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an $(n - 1)$ -dimensional Alexandrov space N . Then by Theorem 2.2, M_i is homeomorphic to Y , and by Theorem 4.19, we have

$$Y = C_0 \times_{\tilde{f}} [-t_0, t_0] / \tilde{f}, \quad N = C_0 / f,$$

where $\tilde{f} = (f, -\text{id})$ is an isometric involution. and the singular locus of the singular I -bundle structure on Y defined by the above form coincides with C_0^1 unless $X \neq X_1$. Later in Lemma 5.5, we show that $\eta_0(C_0^1) = \partial N$.

Assuming that N has non-empty boundary, we begin with construction of singularity models of singular I -fibered spaces around each boundary component of the limit space N .

By Proposition 2.6, each component $\partial_\alpha N$ of ∂N has a collar neighborhood V_α . Let $\varphi : V_\alpha \rightarrow \partial_\alpha N \times [0, 1)$ be a homeomorphism. Let $\pi : Y \rightarrow N$ be the projection. Then I -fiber structure on $\pi^{-1}\varphi^{-1}(\{p\} \times [0, 1))$ is isomorphic to the form

$$R_{t_0} := [0, 1) \times [-t_0, t_0] / (0, y) \sim (0, -y),$$

with the projection $\pi : R_{t_0} \rightarrow [0, 1)$ induced by $(x, y) \rightarrow x$. Therefore $\pi^{-1}(V_\alpha)$ is an R_{t_0} -bundle over $\partial_\alpha N$.

Now we define two singularity model for the singular I -bundle $\pi^{-1}(V_\alpha)$: one is the case when $\pi^{-1}(V_\alpha)$ is a trivial R_{t_0} -bundle over $\partial_\alpha N$, and the other one is the case of non-trivial R_{t_0} -bundle.

Definition 5.1. (1). First, set

$$\mathcal{U}_1(\partial_\alpha N) := \partial_\alpha N \times R_{t_0},$$

and define $\pi : \mathcal{U}_1(\partial_\alpha N) \rightarrow \partial_\alpha N \times [0, 1)$ by $\pi(p, x, y) = (p, x)$ for $(p, x, y) \in \partial_\alpha N \times R_{t_0}$. This gives $\mathcal{U}_1(\partial_\alpha N)$ the structure of a singular I -bundle over $\partial_\alpha N \times [0, 1)$ whose singular locus is $\partial_\alpha N \times 0$. We call this *the product singular I -bundle model* around $\partial_\alpha N$.

(2). For the second model, suppose that $\partial_\alpha N$ admits a double covering space $\rho : P_\alpha \rightarrow \partial_\alpha N$ with the deck transformation φ . Let

$$\mathcal{U}_2(\partial_\alpha N) := (P_\alpha \times R_{t_0}) / \Phi,$$

where Φ is the isometric involution on $P_\alpha \times R_{t_0}$ defined by $\Phi = (\varphi, g)$. Define $\pi : \mathcal{U}_2(\partial_\alpha N) \rightarrow \partial_\alpha N \times [0, 1)$ by $\pi([p, x, y]) = (\rho(p), x)$ for

$(p, x, y) \in P_\alpha \times R_\epsilon$. This gives $\mathcal{U}_2(\partial_\alpha N)$ the structure of a singular I -bundle over $\partial_\alpha N \times [0, 1)$ whose singular locus is $\partial_\alpha N \times 0$. The second model is a twisted one, and is doubly covered by the first model $\mathcal{U}_1(P_\alpha) = P_\alpha \times R_\epsilon$. We call this the *twisted singular I -bundle model* around $\partial_\alpha N$.

Example 5.2. Let us consider the codimension one inradius collapse in Example 3.17. Recall that the limit space Y of \tilde{M}_ϵ is isometric to the form

$$Y = D(E) \times_{\tilde{\phi}} [-t_0, t_0] / (x, t) \sim (r(x), -t),$$

where $r : D(E) \rightarrow D(E)$ denotes the canonical reflection of $D(E)$. If $\pi : Y \rightarrow E$ denotes the projection, then $\pi^{-1}(V)$ is isomorphic to the product singular I -bundle model around ∂E , where V is any collar neighborhood of ∂E .

Example 5.3. Let Q_ϵ denote the space obtained from the disjoint union of two copies of the completion \bar{R}_ϵ of R_ϵ glued along each segment $1 \times [-\epsilon, \epsilon]$ of the boundaries:

$$Q_\epsilon = \bar{R}_\epsilon \amalg_{1 \times [-\epsilon, \epsilon]} \bar{R}_\epsilon.$$

Let $r : Q_\epsilon \rightarrow Q_\epsilon$ be the reflection induced from $(x, y) \rightarrow (x, -y)$. Let $M_\epsilon = (\mathbb{S}^1(1) \times Q_\epsilon) / (z, p) \sim (-z, r(p))$. As $\epsilon \rightarrow 0$, M_ϵ inradius collapses to $\mathbb{S}^1(1/2) \times [0, 2]$. Let $\pi_\epsilon : M_\epsilon \rightarrow \mathbb{S}^1(1/2) \times [0, 2]$ be the projection induced by $[z, (x, y)] \rightarrow (z, x)$. Then both $\pi_\epsilon^{-1}(\mathbb{S}^1(1/2) \times [0, 1])$ and $\pi_\epsilon^{-1}(\mathbb{S}^1(1/2) \times (1, 2])$ are solid Klein bottle and their I -fiber structures are isomorphic to the twisted singular I -bundle model around respective boundary of $\mathbb{S}^1(1/2) \times [0, 2]$.

Theorem 5.4. Let $M_i \in \mathcal{M}(n, \kappa, \lambda, d)$ inradius collapse to an $(n-1)$ -dimensional Alexandrov space N . Then there is a singular I -bundle:

$$I \rightarrow M_i \xrightarrow{\pi} N.$$

More precisely,

- (1) If N has no boundary, then M_i is homeomorphic to a product $N \times I$ or a twisted product $N \tilde{\times} I$;
- (2) If N has non-empty boundary, each component $\partial_\alpha N$ of ∂N has a neighborhood V such that $\pi^{-1}(V)$ is isomorphic to either $\mathcal{U}_1(\partial_\alpha N)$ or $\mathcal{U}_2(\partial_\alpha N)$ as I -fibered spaces;
- (3) If $\pi^{-1}(V)$ is isomorphic to $\mathcal{U}_1(\partial_\alpha N)$ for some component $\partial_\alpha N$, then M_i is homeomorphic to $D(N) \times [-1, 1] / (x, t) \sim (r(x), -t)$, where r is the canonical reflection of the double $D(N)$.

Recall that

$$Y = C_0 \times_{\tilde{\phi}} [-t_0, t_0] / \tilde{f},$$

where $\tilde{f} = (f, -\text{id})$, C_0 and Y are the noncollapsing limit of $(\partial M_i)^{\text{int}}$ and \tilde{M}_i respectively. Therefore both C_0 and $Y \setminus C_{t_0}$ are smoothable spaces in the sense of [12]. See also Remark 2.11.

Let $F \subset C_0$ denote the fixed point set of the isometry $f : C_0 \rightarrow C_0$. By Proposition 2.5 and Theorem 2.3, $\eta_0(F)$ is an extremal subset of N and it has a topological stratification.

Lemma 5.5. $\eta_0(F)$ coincides with ∂N if f is not the identity.

We postpone the proof of Lemma 5.5 for a moment.

Proof of Theorem 5.4. (1) By Lemma 5.5, if N has no boundary, F is empty, and therefore either $N = N_1$ or $N = N_2$. If $N = N_1$, then $C_0 = N$ and Y is homeomorphic to $N \times I$. If $N = N_2$, then $N = C_0/f$ has no boundary, and Y is homeomorphic to either $N \times I$ or $C_0 \times [-1, 1]/(x, t) \sim (f(x), -t)$ which is a twisted I bundle over N .

(2) Suppose N has non-empty boundary. Note that

$$N_1 = \eta_0(F).$$

By Proposition 2.6, each component $\partial_\alpha N$ of ∂N has a collar neighborhood V_α . Let $\varphi : V_\alpha \rightarrow \partial_\alpha N \times [0, 1]$ be a homeomorphism. Let $\pi : Y \rightarrow N$ be the projection. By the I -fiber structure of Y , $\pi^{-1}(\varphi^{-1}(x \times [0, 1]))$ is canonically homeomorphic to R_{t_0} . In particular $\pi^{-1}(V_\alpha)$ is an R_{t_0} -bundle over $\partial_\alpha N$. If this bundle is trivial, $\pi^{-1}(V_\alpha)$ is isomorphic to the product singular I -bundle structure $\mathcal{U}_1(\partial_\alpha N) = \partial_\alpha N \times R_{t_0}$.

Suppose that this bundle is nontrivial, and let P_α be the boundary of $\pi^{-1}(\varphi^{-1}(\partial_\alpha N \times \{1/2\}))$, which is a double covering of ∂N_α . Let $\Phi = (\varphi, g)$, and $\rho : P_\alpha \rightarrow \partial_\alpha N$ the projection.

Lemma 5.6. $\pi^{-1}(V_\alpha)$ is isomorphic to the twisted singular I -bundle structure $\mathcal{U}_2(\partial_\alpha N) = (P_\alpha \times R_{t_0})/\Phi$.

Proof. Note that

$$\begin{aligned} \mathcal{U}_2(\partial_\alpha N) &:= (P_\alpha \times R_{t_0})/(p, x, y) \sim (\varphi(p), x, -y), \\ \pi^{-1}(V_\alpha) &= \pi^{-1}\varphi^{-1}(\partial_\alpha N \times [0, 1]). \end{aligned}$$

We define a map $\Psi : \mathcal{U}_2(\partial_\alpha N) \rightarrow \pi^{-1}(V_\alpha)$ as follows: Note that for each $(p, x) \in P_\alpha \times [0, 1]$, $\{p, \varphi(p)\}$ can be identified with the boundary of the I -fiber $I_{\rho(p), x} := \pi^{-1}\varphi^{-1}(\rho(p) \times \{x\})$. Define $\Psi(p, x, y)$, $-t_0 \leq y \leq t_0$, be the arc on the fiber $I_{\rho(p), x}$ from p to $\varphi(p)$. It is easy to see that $\Psi : \mathcal{U}_2(\partial_\alpha N) \rightarrow \pi^{-1}(V_\alpha)$ gives an isomorphism between I fibered spaces. \square

(3) Put $\text{int}N := N \setminus \partial N$ for simplicity.

Assertion 5.7. There is an isometric imbedding $g : N \rightarrow C_0$ such that $\eta_0 \circ g = 1_N$.

Proof. Set $F_\alpha := \eta_0^{-1}(\partial_\alpha N)$. From the assumption, we may assume that F_α is two-sided in the sense that the complement of F_α in some connected neighborhood of it is disconnected. Thus there is a connected neighborhood V_α of $\partial_\alpha N$ in $\text{int}N$ for which there is an isometric imbedding $g_\alpha : V_\alpha \rightarrow C_0 \setminus F$ such that $\eta_0 \circ g_\alpha = 1_{V_\alpha}$.

Let W be the maximal connected open subset of $\text{int}N$ for which there is an isometric imbedding $g_0 : W \rightarrow C_0 \setminus F$ such that $\eta_0 \circ g_0 = 1_W$ and $g_0(W) \supset g_\alpha(V_\alpha)$. We only have to show that $W = \text{int}N$. Otherwise, there is a point $x \in \partial W \cap \text{int}N$. Take a connected neighborhood W_x of x in $\text{int}N$ such that $\eta_0^{-1}(W_x)$ is a disjoint union of open sets U_1 and U_2 such that $\eta_0 : U_i \rightarrow W_x$ is an isometry for $i = 1, 2$. Obviously one of U_i , say U_1 , meets $g_0(W)$ and the other does not. We extend g_0 to $g_1 : W \cup W_x \rightarrow C_0 \setminus F$ by requiring $g_1|_{W_x} = \eta_0^{-1} : W_x \rightarrow U_1$. Since g_1 is an isometric imbedding, this is a contradiction to the maximality of W .

Thus we have an isometric imbedding $g_0 : \text{int}N \rightarrow C_0 \setminus F$. Since $\text{int}N$ is convex and η_0 is 1-Lipschitz, g_0 preserves the distance. It follows that g_0 extends to an isometric imbedding $g : N \rightarrow C_0$ which preserves distance. \square

Assertion 5.7 shows that every component of F is two-sided. It follows that $C_0 = D(N)$, and that f is the reflection of the double $D(N)$. This completes the proof of Theorem 5.4 \square

Proof of Lemma 5.5. Obviously $\partial N \subset \eta_0(F)$. Suppose that $\eta_0(F) \cap (\text{int}N)$ is not empty.

Sublemma 5.8. $\dim(\eta_0(F) \cap \text{int}N) \leq m - 2$, where $m := \dim N$.

Proof. If $\dim(\eta_0(F) \cap \text{int}N) = m - 1$, then the top-dimensional strata S of $\eta_0(F) \cap \text{int}N$ is a topological $(m - 1)$ -manifold, and therefore it meets the m -dimensional strata of N because $N^{\text{sing}} \cap \text{int}N$ has codimension ≥ 2 (Theorem 2.1). Take $p \in \eta_0^{-1}(S)$. It is now easy to see that f is the reflection with respect to $\eta_0^{-1}(S)$ in a small neighborhood of p . It follows that S is a subset of ∂N , contradiction to the hypothesis. \square

Take a point $x = \eta_0(p) \in \eta_0(F) \cap \text{int}N$, and consider the directional derivatives $f_* : \Sigma_p(C_0) \rightarrow \Sigma_p(C_0)$ of f at p which is again an isometric involution with fixed point set

$$F_* := \Sigma_p(F)$$

By Corollary 2.4 and Sublemma 5.8, $\dim F_* \leq m - 3$ while $\dim \Sigma_p(C_0) = m - 1$. Repeating this we have a finite sequence of directional derivatives of f , $f_* \dots$, each of which is an isometric involution:

$$f_{*k} : \Sigma_{*k}(C_0) \rightarrow \Sigma_{*k}(C_0),$$

where $\Sigma_{*k}(C_0)$ denotes a k iterated space of directions,

$$\Sigma_{*k}(C_0) = \Sigma_{\xi_{k-1}}(\dots(\Sigma_{\xi_1}(\Sigma_p(C_0))\dots)),$$

and ξ_i is taken from the fixed point set of the iterated directional derivatives:

$$\xi_1 \in \Sigma_p(F), \xi_2 \in \Sigma_{\xi_1}(F_*), \dots, \xi_k \in \Sigma_{\xi_{k-1}}(F_{*(k-1)}),$$

and F_{*i} denotes the fixed point set of $f_{*i} : \Sigma_{*i}(C_0) \rightarrow \Sigma_{*i}(C_0)$ which coincides with $F_{*i} = \Sigma_{\xi_{i-1}}(F_{*(i-1)})$.

Note that the iterated space of directions $\Sigma_{*k}(C_0)$ has dimension $m - k$, and the iterated fixed point set $F_{*k} \subset \Sigma_{*k}(C_0)$ has dimension $\leq m - k - 2$. It follows that for some $k \leq m - 2$, F_{*k} becomes a finite set. It follows that for any $\xi_{k+1} \in F_{*k}$,

$$f_{*(k+1)} : \Sigma_{\xi_{k+1}}(\Sigma_{*k}(C_0)) \rightarrow \Sigma_{\xi_{k+1}}(\Sigma_{*k}(C_0))$$

has no fixed points. Put

$$D := C_0 \times_{\tilde{\phi}} [-t_0, t_0],$$

and let \tilde{f} be an isometric involution on D defined by $\tilde{f} = (f, -\text{id})$. From Theorem 4.19,

$$Y = D/\tilde{f}.$$

Let $x = \eta_0(p)$, $p = (p, 0)$, $\xi_i \in \Sigma_{\xi_{i-1}}(F_{*(i-1)})$, $1 \leq i \leq k + 1$, be as above. Note that

$$\Sigma_x(Y) = \Sigma_p(D)/\tilde{f}_*, \quad \Sigma_x(X) = \Sigma_p(C_0)/f_*.$$

Let $\zeta_1 \in \Sigma_x(\eta_0(F)) \subset \Sigma_x(X) \subset \Sigma_x(Y)$ be the element corresponding to $\xi_1 \in \Sigma_p(F) \subset \Sigma_p(C_0) \subset \Sigma_p(D)$. Note that

$$\Sigma_p(D) = \{\xi_{\pm}\} * \Sigma_p(C_0)$$

and $\tilde{f}_* = (f_*, -\text{id})$ interchanges ξ_+ and ξ_- and preserves $\Sigma_p(C_0)$. Next consider

$$\Sigma_{\zeta_1}(\Sigma_x(Y)) = \Sigma_{\xi_1}(\Sigma_p(D))/\tilde{f}_{**},$$

where \tilde{f}_{**} denotes the directional derivatives of \tilde{f}_* at ζ_1 . Note that $\Sigma_{\xi_1}(\Sigma_p(D))$ is still isometric to $\{\xi_{\pm}\} * \Sigma_{\xi_1}(\Sigma_p(C_0))$ and $\tilde{f}_{**} = (f_{**}, -\text{id})$ interchanges ξ_+ and ξ_- and preserves $\Sigma_{\xi_1}(\Sigma_p(C_0))$. Similarly and finally we consider

$$(5.12) \quad \Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y)) = \Sigma_{\xi_{k+1}}(\Sigma_{*k}(D))/\tilde{f}_{*k+1},$$

where $\zeta_{k+1} \in \Sigma_{*k}(Y)$ is the element corresponding to $\xi_{k+1} \in \Sigma_{*k}(D)$, and $\tilde{f}_{*k+1} = (f_{*k+1}, -\text{id})$ freely acts on $\Sigma_{\xi_{k+1}}(\Sigma_{*k}(D))$. Recall that

$$\ell := \dim \Sigma_{\xi_{k+1}}(\Sigma_{*k}(D)) = m - k \geq 2.$$

Note that the iterated spaces of directions of the smoothable spaces $Y \setminus \partial C_{t_0}$ must be all homeomorphic to spheres (Theorem 2.7). However (5.12) shows that $\Sigma_{\zeta_{k+1}}(\Sigma_{*k}(Y))$ is homeomorphic to a quotient $\mathbb{S}^{\ell}/\mathbb{Z}_2$ for $\ell \geq 2$ by a free \mathbb{Z}_2 -action, which is a contradiction. This completes the proof of Lemma 5.5. \square

5.2. Inradius collapse to almost regular spaces. Next we consider the case where M_i inradius collapses to almost regular Alexandrov space N . The idea of using an equivariant fibration-capping theorem in [36] was inspired by a recent work [19].

First we recall this theorem. Let X be a k -dimensional complete Alexandrov space with curvature $\geq \kappa$ possibly non-empty boundary. We denote by $D(X)$ the double of X , which is also an Alexandrov space with curvature $\geq \kappa$. (see [22]). By definition, $D(X) = X \cup X^*$ glued along their boundaries, where X^* is another copy of X .

A (k, δ) -strainer $\{(a_i, b_i)\}$ of $D(X)$ at $p \in X$ is called *admissible* if $a_i \in X$, $b_j \in X$ for every $1 \leq i \leq k$, $1 \leq j \leq k-1$ (clearly, $b_k \in X^*$ if $p \in \partial X$ for instance). Let $R_\delta^D(X)$ denote the set of points of X at which there are admissible (k, δ) -strainers. It has the structure of a Lipschitz k -manifold with boundary. Note that every point of $R_\delta^D(X) \cap \partial X$ has a small neighborhood in X almost isometric to an open subset of the half space \mathbb{R}_+^k for small δ .

If Y is a closed domain of $R_\delta^D(X)$, then the δ_D -strain radius of Y is defined as the infimum of positive numbers ℓ such that there exists an admissible (k, δ) -strainer of length $\geq \ell$ at every point in Y , denoted by $\delta_D\text{-str.rad}(Y)$.

For a small $\nu > 0$, we put

$$Y_\nu := \{x \in Y \mid d(\partial X, x) \geq \nu\}.$$

We use the following special notations:

$$\partial_0 Y_\nu := Y_\nu \cap \{d_{\partial X} = \nu\}, \quad \text{int}_0 Y_\nu := Y_\nu - \partial_0 Y_\nu.$$

Let M^n be another n -dimensional complete Alexandrov space with curvature $\geq \kappa$ having no boundary. Let $R_\delta(M)$ denote the set of all (n, δ) -strained points of M .

A surjective map $f : M \rightarrow X$ is called an ϵ -almost Lipschitz submersion if

- (1) it is an ϵ -approximation;
- (2) for every $p, q \in M$

$$\left| \frac{d(f(p), f(q))}{d(p, q)} - \sin \theta_{p,q} \right| < \epsilon,$$

where $\theta_{p,q}$ denotes the infimum of $\angle qpx$ when x runs over $f^{-1}(f(p))$.

Now let a Lie group G act on M^n and X as isometries. Let

$$d_{e.GH}((M, G), (X, G))$$

denote the equivariant Gromov-Hausdorff distance as defined in Section 2.1. We need to assume the following on the existence of slice for G -orbits:

Assumption 5.9. For each $p \in X$, there is a *slice* L_p at p . Namely $U_p := GL_p$ provides a G -invariant tubular neighborhood of Gp which is G -isomorphic to $G \times_{G_p} L_p$.

Obviously Assumption 5.9 is automatically satisfied if G is discrete. By [11], Assumption 5.9 also holds true if G is compact.

Theorem 5.10 (Equivariant Fibration-Capping Theorem([36], Thm 18.9)). *Let X and G be as above such that X/G is compact. Given k and $\mu > 0$ there exist positive numbers $\delta = \delta_k$, $\epsilon_{X,G}(\mu)$ and $\nu = \nu_{X,G}(\mu)$ satisfying the following : Suppose $X = R_\delta^D(X)$ and $\delta_D\text{-str.rad}(X) > \mu$. Suppose $M = R_{\delta_n}(M)$ and $d_{eGH}((M,G), (X,G)) < \epsilon$ for some $\epsilon \leq \epsilon_{X,G}(\mu)$. Then there exists a G -invariant decomposition*

$$M = M_{\text{int}} \cup M_{\text{cap}}$$

of M into two closed domains glued along their boundaries, and a G -equivariant Lipschitz map $f : M \rightarrow X_\nu$ such that

- (1) M_{int} is the closure of $f^{-1}(\text{int}_0 X_\nu)$, and $M_{\text{cap}} = f^{-1}(\partial_0 X_\nu)$;
- (2) the restrictions $f|_{M_{\text{int}}} : M_{\text{int}} \rightarrow X_\nu$ and $f|_{M_{\text{cap}}} : M_{\text{cap}} \rightarrow \partial_0 X_\nu$ are
 - (a) locally trivial fiber bundles;
 - (b) $\tau(\delta, \nu, \epsilon/\nu)$ -Lipschitz submersions.

Here, $\tau(\epsilon_1, \dots, \epsilon_k)$ denotes a function depending on a priori constants and ϵ_i satisfying

$$\lim_{\epsilon_i \rightarrow 0} \tau(\epsilon_1, \dots, \epsilon_k) = 0.$$

Remark 5.11. If X has no boundary, then X_ν is replaced by X , $M_{\text{cap}} = \emptyset$ and $M = N$ in the statement above.

We go back to the situation of Theorem 1.4. Assume that M_i inradius collapses to an almost regular Alexandrov space N . Let us consider the double and the partial double of \tilde{M}_i and Y respectively

$$D(\tilde{M}_i) := \tilde{M}_i \amalg_{\partial \tilde{M}_i} \tilde{M}_i, \quad W := Y \amalg_{C_{t_0}} Y,$$

where two copies of Y are glued along C_{t_0} . From Perelman's result [22], both $D(\tilde{M}_i)$ and W are Alexandrov space. Note that both $D(\tilde{M}_i)$ and W admit canonical isometric \mathbb{Z}_2 actions by the reflections.

The proof of the following lemma is standard, and hence omitted.

Lemma 5.12. *$(D(\tilde{M}_i), \mathbb{Z}_2)$ converges to (W, \mathbb{Z}_2) with respect to the equivariant Gromov-Hausdorff convergence.*

Proof of Theorem 1.4. By Lemma 5.12, for any $\epsilon > 0$, if i is large,

$$d_{eGH}((D(\tilde{M}_i), \mathbb{Z}_2), (W, \mathbb{Z}_2)) < \epsilon.$$

By Theorem 4.19, Y is almost regular possibly with almost regular boundary. Hence, $W = R_\delta^D(W)$ and $\delta_D\text{-str.rad}(W) > \mu$ for some $\mu > 0$. Thus by Theorem 5.10 and its remark, there exists a \mathbb{Z}_2 -equivariant capping fibration

$$\tilde{f}_i : D(\tilde{M}_i) \rightarrow W_\nu,$$

where

$$W_\nu = \{x \in W \mid d(x, \partial W) \geq \nu\}.$$

Notice that W_ν is homeomorphic to W because of the form of Y . Obviously, \tilde{f}_i induces a map $f_i : \tilde{M}_i \rightarrow Y$. By the remark after Corollary 4.21, $\eta_0 : C_0 \rightarrow X$ is either an isometry or a locally isometric double covering.

Case (a). If $\eta_0 : C_0 \rightarrow X$ is a double covering, then $C_{t_0} = \partial Y$. Hence W has no boundary. Thus in this case, $f_i : \tilde{M}_i \rightarrow Y$ is a fiber bundle with fiber F_i which are closed almost nonnegatively curved manifolds. Since Y is an I -bundle over N by Theorem 4.19, \tilde{M}_i and hence M_i is an $F_i \times I$ -bundle over N .

Case (b). If $\eta_0 : C_0 \rightarrow X$ is an isometry, then Y is isometric to $N \times_\phi [0, t_0]$, and therefore ∂Y consists of $\eta(C_0) = X$ and $\eta(C_{t_0})$. Thus ∂W consists two copies of $\eta_0(C_0)$. Therefore by Theorem 5.10, there exists a \mathbb{Z}_2 -invariant decomposition

$$(5.13) \quad D(\tilde{M}_i) = (D(\tilde{M}_i))_{\text{int}} \cup (D(\tilde{M}_i))_{\text{cap}},$$

of $D(\tilde{M}_i)$ into two closed domains glued along their boundaries such that

- (1) $(D(\tilde{M}_i))_{\text{int}}$ is the closure of $\tilde{f}_i^{-1}(\text{int}_0 W_\nu)$, and $(D(\tilde{M}_i))_{\text{cap}} = \tilde{f}_i^{-1}(\partial_0 W_\nu)$;
- (2) $\tilde{f}_i|_{(D(\tilde{M}_i))_{\text{int}}} : (D(\tilde{M}_i))_{\text{int}} \rightarrow W_\nu$, $\tilde{f}_i|_{(D(\tilde{M}_i))_{\text{cap}}} : (D(\tilde{M}_i))_{\text{cap}} \rightarrow \partial_0 W_\nu$ are locally trivial fiber bundles,

where

$$\partial_0 W_\nu := \{x \in W \mid d(x, \partial W) = \nu\}, \quad \text{int}_0 W_\nu := W_\nu \setminus \partial_0 W_\nu.$$

Since (5.13) is \mathbb{Z}_2 -invariant, it induces a decomposition

$$\tilde{M}_i = (\tilde{M}_i)_{\text{int}} \cup (\tilde{M}_i)_{\text{cap}}.$$

Since \tilde{f}_i is \mathbb{Z}_2 -equivariant, these fibrations induce fibrations

$$\begin{aligned} F_i &\longrightarrow (\tilde{M}_i)_{\text{int}} \longrightarrow Y_\nu, \\ \text{Cap}_i &\longrightarrow (\tilde{M}_i)_{\text{cap}} \longrightarrow \partial_0 Y_\nu. \end{aligned}$$

From construction, ∂Cap_i is homeomorphic to F_i . Note that every cylindrical geodesic in the warped cylinder $C_i \subset \tilde{M}_i$ is almost perpendicular to the fibers ([34], [35]). This implies that $(\tilde{M}_i)_{\text{int}}$ is homeomorphic to $\partial(\tilde{M}_i)_{\text{int}} \times [0, 1]$, and therefore \tilde{M}_i and hence M_i is homeomorphic to $(\tilde{M}_i)_{\text{cap}}$. Noting $\partial_0 Y_\nu$ is homeomorphic to N , we obtain a fiber bundle

$$\text{Cap}_i \longrightarrow M_i \longrightarrow N.$$

This completes the proof. \square

6. THE CASE OF UNBOUNDED DIAMETERS

In this section we provide the proof of Theorem 1.6.

Remark 6.1. Theorem 1.6 (1) was stated in [33], Theorem 5, where the following argument was employed: If $k \geq 3$ and if $p \in M$ is the furthest point from ∂M , then $B(p, r)$, $r = \text{inrad}(M)$, touches ∂M at least three points. However it seems to the authors that this is unclear.

Let $\mathcal{M}(n, \kappa, \lambda)$ denote the set of all isometry classes of n -dimensional complete Riemannian manifolds M satisfying

$$K_M \geq \kappa, \quad |\Pi_{\partial M}| \leq \kappa.$$

Let $\tilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$ denote the set of all (\tilde{M}, M, p) with $M \in \mathcal{M}(n, \kappa, \lambda)$ and $p \in \partial M$.

We denote by

$$\partial_0 \mathcal{M}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$$

the set of all pointed Gromov-Hausdorff limit spaces (Y, X, x) of sequences (\tilde{M}_i, M_i, p_i) in $\tilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$ with $\text{inrad}(M_i) \rightarrow 0$. From now, (\tilde{M}_i, M_i, p_i) and (\tilde{M}, M, p) are always assumed to be elements in $\tilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$.

We first remark

Lemma 6.2. *Let (\tilde{M}_i, M_i, p_i) converges to (Y, X, x) in $\partial_0 \mathcal{M}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$ with $\text{inrad}(M_i) \rightarrow 0$. Then all the structure results for the limit spaces in Section 4.1 and Lemma 4.8 still holds for (Y, X) .*

Proof. This is because all the argument there are local. \square

However in the proof of Corollary 4.9, the compactness of C_0 is essentially used. To improve this proof for the noncompact C_0 , we need to establish the following result, which is a weaker form of Lemma 4.17.

Proposition 6.3. *For every $x, y \in X$ and $p \in C_0$ with $\eta_0(p) = x$, there exists a point $q \in C_0$ such that $\eta_0(q) = y$ and $|p, q| = |x, y|_{X^{\text{int}}}$.*

Proof. Let $c : [0, \ell] \rightarrow X$ be a unit speed minimal geodesic joining x to y , and let $v := \dot{c}(0) \in \Sigma_x(X)$. By Lemmas 4.2 and 4.3, $\angle(\xi_+, v) = \pi/2$. Let $\xi \in \Sigma_x(Y)$ be an element satisfying (4.3) or (4.7) together with $\angle(\xi_+, \xi) = \pi/4$. We may assume that there is a geodesic γ from x in the direction ξ . Consider $\tilde{\gamma} := \eta^{-1}(\gamma)$ and $\tilde{\sigma} := \tilde{\pi}(\tilde{\gamma})$. Let $\sigma := \pi(\gamma) = \eta_0(\tilde{\sigma})$, and σ define a direction \hat{v} . As was shown in the proof of Lemma 4.2, \hat{v} also satisfies the same equation (4.3) as v . This shows that $\hat{v} = v$ and σ is infinitesimally minimizing because it has a definite direction at x . Therefore for every $\epsilon > 0$, setting $x_1 := \sigma(t_1)$ for small enough $t_1 > 0$, we have

$$|y, x_1| \leq |x, y| - (1 - \epsilon)L(\sigma_1),$$

where $\sigma_1 = \sigma|_{[0, t_1]}$. We repeat this argument for a minimal geodesic $c_1 : [0, \ell_1] \rightarrow X$ joining x_1 to y , and finally we have an infinite sequence

of points $\{x_i\}_{i=1}^\infty$ and Lipschitz curves $\{\sigma_i\}_{i=1}^\infty$ and $\{\tilde{\sigma}_i\}_{i=1}^\infty$ joining x_{i-1} to x_i and \tilde{x}_{i-1} to \tilde{x}_i respectively such that

- (1) $\eta_0(\tilde{\sigma}_i) = \sigma_i$;
- (2) $|y, x_k| \leq |x, y| - (1 - \epsilon) \sum_{i=1}^k L(\sigma_i)$ for each $1 \leq k < \infty$;
- (3) $\lim x_k = y$.

Let the curves σ_ϵ and $\tilde{\sigma}_\epsilon$ be defined by the union of those σ_i and $\tilde{\sigma}_i$ respectively. It follows that σ_ϵ is an almost minimizing curve joining x to y . Passing to a subsequence we may assume that σ_ϵ and $\tilde{\sigma}_\epsilon$ converge to curves σ_∞ and $\tilde{\sigma}_\infty$ respectively satisfying $\eta_0(\tilde{\sigma}_\infty) = \sigma_\infty$. Note that both σ_∞ and $\tilde{\sigma}_\infty$ are minimizing since η_0 is 1-Lipschitz and preserving length by Proposition 4.5. Thus we have a required point q as the endpoint of $\tilde{\sigma}_\infty$ different from p . \square

By Proposition 6.3, we can improve the proof of Corollary 4.9 without assuming the compactness. Thus Corollary 4.9 holds for the present case, too. We also see that all the results in Section 4 still holds true except Corollary 4.11. In particular we have

Theorem 6.4. *Let a sequence of pointed complete Riemannian manifolds (M_i, p_i) in $\mathcal{M}(n, \kappa, \lambda)$ inradius collapse to a pointed length space (N, q) with respect to the pointed Gromov-Hausdorff convergence. Then N is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, where $c(\kappa, \lambda)$ is a constant depending only on κ and λ .*

To have Corollary 4.11 in the case when Y is noncompact is the main purpose of the rest of this section.

We introduce a more refined version of the pointed Gromov-Hausdorff convergence. Let $\iota_{\partial M} : (\partial M)^{\text{int}} \rightarrow (\partial M)^{\text{ext}}$ be the canonical map, where $(\partial M)^{\text{ext}}$ is equipped with the exterior metric in M . Let $\omega_M : M \rightarrow \partial M$ be a nearest point map (compare Proposition 3.9).

Definition 6.5. For (\tilde{M}, M, p) and $(Y, X, x) \in \partial_0 \mathcal{MM}(n, \kappa, \lambda)_{\text{pt}}$ with

$$(6.14) \quad Y = X \bigcup_{\eta_0} C_0 \times_\phi [0, t_0],$$

we define the pointed Gromov-Hausdorff distance

$$d_{pGH}((\tilde{M}, M, p), (Y, X, x))$$

as the infimum of those $\delta > 0$ such that

- (1) there exists a component-wise δ -approximation $\psi : B^{\tilde{M}}(p, 1/\delta) \cap (\partial M)^{\text{int}} \rightarrow B^Y(x, 1/\delta) \cap C_0^{\text{int}}$;
- (2) the map $\varphi : B^{M^{\text{int}}}(p, 1/\delta) \rightarrow B^{X^{\text{int}}}(x, 1/\delta)$ defined by

$$\varphi = \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M$$

is a δ -approximation;

(3) the map $\Phi : B^{\tilde{M}}(p, 1/\delta) \rightarrow B^Y(x, 1/\delta)$ defined by

$$\Phi(q) = \begin{cases} \varphi(q), & q \in B^{\tilde{M}}(p, 1/\delta) \cap M \\ (\eta_0 \circ \psi \circ \iota_{\partial M}^{-1}(q_1), t), & q = (q_1, t) \in B^{\tilde{M}}(p, 1/\delta) \cap \partial M \times [0, t_0]. \end{cases}$$

is a δ -approximation.

This definition is justified by the following lemma.

Lemma 6.6. *Let $(\tilde{M}_i, M_i, p_i) \in \tilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$ converge to (Y, X, x) in $\partial_0 \mathcal{M}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$ for the pointed Gromov-Hausdorff topology. Then there exists a component-wise δ_i -approximation*

$$\psi_i : B^{\tilde{M}_i}(p_i, 1/\delta_i) \cap (\partial M_i)^{\text{int}} \rightarrow B^Y(x, 1/\delta_i) \cap C_0$$

with $\lim \delta_i = 0$ such that the maps

$$\varphi_i : B^{M_i^{\text{int}}}(p_i, 1/\delta_i) \rightarrow B^{X^{\text{int}}}(x, 1/\delta_i), \Phi_i : B^{\tilde{M}_i}(p_i, 1/\delta_i) \rightarrow B^Y(x, 1/\delta_i)$$

defined as in Definition 6.5 via ψ_i are δ'_i -approximations with $\lim \delta'_i = 0$.

Proof. Let $\lambda_i : B^{\tilde{M}_i}(p_i, 1/\epsilon_i) \rightarrow B^Y(x, 1/\epsilon_i)$ be an ϵ_i -approximation with $\lim \epsilon_i = 0$. When it is restricted to the boundary, it provides a component-wise ϵ_i -approximation $\lambda'_i : B^{\tilde{M}_i}(p_i, 1/\epsilon_i) \cap \partial M_i \rightarrow B^Y(x, 1/\epsilon_i) \cap C_{t_0}$. Since $\partial \tilde{M}_i$ and C_{t_0} are totally geodesic and $\phi(t_0)$ -homothetic to $(\partial M_i)^{\text{int}}$ and C_0 respectively, λ'_i gives a component-wise $\epsilon_i/\phi(t_0)$ -approximation $\psi_i : B^{\tilde{M}_i}(p_i, 1/\epsilon_i) \cap (\partial M_i)^{\text{int}} \rightarrow B^Y(x, 1/\epsilon_i) \cap C_0$. Now the conclusion follows in a way similar to Proposition 3.9. \square

Lemma 6.7. *For each $\delta > 0$ there exists a positive number $\epsilon = \epsilon(\delta)$ such that if (M, p) in $\mathcal{M}(n, \kappa, \lambda)$ satisfies $\text{inrad}(M) < \epsilon$, then*

$$d_{pGH}((\tilde{M}, M, p), (Y, X, x)) < \delta,$$

for some (Y, X, x) contained in $\partial_0 \mathcal{M}\mathcal{M}(n, \kappa, \lambda)$.

Proof. Lemma 6.7 follows from Lemma 6.6 and the precompactness of $\tilde{\mathcal{M}}\mathcal{M}(n, \kappa, \lambda)_{\text{pt}}$ combined with a contradiction argument. \square

If $(Y, X, x) \in \partial_0 \mathcal{M}\mathcal{M}(n, \kappa, \lambda)$ satisfies the conclusion of Lemma 6.7, we call it a δ -limit of (\tilde{M}, M, p) , which is also denoted by $\mathcal{Y}(M, p)$ for simplicity:

$$\mathcal{Y}(M, p) = (Y, X, x).$$

Definition 6.8. Let $(Y, X, x) \in \partial_0 \mathcal{M}\mathcal{M}(n, \kappa, \lambda)$ and $y \in X$. We call y a *single point* (resp a *double point*) if $\#\eta_0^{-1}(y) = 1$ (resp. $\#\eta_0^{-1}(y) = 2$). We say that (Y, X, x) is *single* (resp *double*) if every element of X is *single* (resp *double*). If (Y, X, x) neither *single* nor *double*, it is called *mixed*. We also say that (Y, X, x) is *single* (resp. *double*) in scale R if every element of $X \cap B^Y(x, R)$ is *single* (resp. *double*). If (Y, X, x) is neither *single* nor *double* in scale R , it is called *mixed* in scale R .

From now on, to prove Theorem 1.6, we analyze the local structure of ∂M about the connectedness when $\text{inrad}(M) < \epsilon$. By Lemma 6.7, for any $p \in M$, there exists a δ -limit $\mathcal{Y}(M, p) = (Y, X, x)$ together with

- (1) a δ -approximation $\psi : (\partial M)^{\text{int}} \cap B^M(p, R) \rightarrow C_0(p, R)$, where $C_0(p, R)$ is a closed domain in C_0^{int} ;
- (2) a δ -approximation $\varphi := \eta_0 \circ \psi \circ \iota_{\partial M}^{-1} \circ \omega_M : B^M(p, R) \rightarrow B^{X^{\text{int}}}(x, R)$.

We shall use those δ -approximations in the proofs of Lemmas 6.9, 6.10 and 6.11 below.

Lemma 6.9. *For any $R > 0$ there exists $\delta_0 < 1/R$ satisfying the following: For every $0 < \delta \leq \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 6.7. Then for M in $\mathcal{M}(n, \kappa, \lambda)$ with $\text{inrad}(M) < \epsilon$, if some δ -limit $\mathcal{Y}(M, p)$ is single in scale R for some $p \in M$, then every $p_1, p_2 \in \partial M \cap B^{\tilde{M}}(p, R)$ can be joined by a curve in ∂M of length $\leq |p_1, p_2|_M + 2\delta$.*

Proof. Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be δ -approximations as above. Put $x_i := \varphi(p_i) \in X$, $i = 1, 2$. Take $\tilde{x}_i \in C_0$ such that $\eta_0(\tilde{x}_i) = x_i$. Lemma 4.14 shows $|\tilde{x}_1, \tilde{x}_2| = |x_1, x_2|$. Since ψ is a δ -approximation, we have

$$|p_1, p_2|_{\partial M} < |\tilde{x}_1, \tilde{x}_2| + \delta = |x_1, x_2| + \delta < |p_1, p_2|_M + 2\delta.$$

□

Lemma 6.10. *For any $R > 0$ there exists $\delta_0 < 1/R$ satisfying the following: For every $0 < \delta \leq \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 6.7. Then for M in $\mathcal{M}(n, \kappa, \lambda)$ with $\text{inrad}(M) < \epsilon$, if a δ -limit $\mathcal{Y}(M, p)$ is double in scale R for some $p \in M$, then there exists a point $p' \in M$ satisfying*

- (1) $|p, p'|_M < \delta$;
- (2) every $q \in \partial M \cap B^{\tilde{M}}(p, R)$ can be joined to p or p' by a curve in ∂M of length $\leq |p, q|_M + 3\delta$.

Proof. Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be δ -approximations as above. Set $x := \varphi(p)$, $y := \varphi(q)$. Since (Y, X, x) is double in scale R , we can put $\{\tilde{x}_1, \tilde{x}_2\} := \eta_0^{-1}(x)$ and $\{\tilde{y}_1, \tilde{y}_2\} := \eta_0^{-1}(y)$. Let $\gamma : [0, 1] \rightarrow X$ be a minimal geodesic joining x to y . From Lemma 4.17, there are lifts $\tilde{\gamma}_i : [0, 1] \rightarrow C_0$ of γ starting from \tilde{x}_i , where we may assume $\tilde{\gamma}(1) = \tilde{y}_i$ and $\tilde{x}_1 = \psi(p)$. Since ψ is a δ -approximation, if $\psi(q) = \tilde{y}_1$, then

$$|p, q|_{(\partial M)^{\text{int}}} < |\tilde{x}_1, \tilde{y}_1| + \delta = |x, y| + \delta < |p, q|_{\partial M^{\text{ext}}} + 2\delta$$

Similarly, if $\psi(q) = \tilde{y}_2$, then putting $p' := \psi^{-1}(\tilde{x}_2)$, we have $|p', q|_{(\partial M)^{\text{int}}} < |p, q|_M + 3\delta$. This completes the proof. □

Lemma 6.11. *For any $R > 0$ there exists $\delta_0 < 1/R$ satisfying the following: For every $0 < \delta \leq \delta_0$, let $\epsilon = \epsilon(\delta) > 0$ be as in Lemma 6.7. Then for M in $\mathcal{M}(n, \kappa, \lambda)$ with $\text{inrad}(M) < \epsilon$, if a δ -limit $\mathcal{Y}(M, p)$*

is mixed in scale R for some $p \in M$, then there exists a point $p_0 \in \partial M \cap B^{\tilde{M}}(p, R)$ such that every point q in $\partial M \cap B^{\tilde{M}}(p, R)$ can be joined to p_0 by a minimal geodesic in ∂M of length $|p_0, q|_M + 2\delta$.

Proof. Let $(Y, X, x) := \mathcal{Y}(M, p)$, and ψ, φ be δ -approximations as above. Let $x_0 \in X$ be a single point with $|x, x_0| \leq R$, and take $\tilde{x}_0 \in C_0$ and $p_0 \in \partial M$ such that $\eta_0(\tilde{x}_0) = x_0$ and $|\psi(p_0), \tilde{x}_0| < \delta$. Let $\gamma : [0, 1] \rightarrow X$ be a minimal geodesic from x_0 to $\varphi(q)$. Since $\tilde{x}_0 \in C_0^1$, there is a unique minimal geodesic $\tilde{\gamma} : [0, 1] \rightarrow C_0$ from \tilde{x}_0 to $\psi(q)$ with $\eta_0 \circ \tilde{\gamma} = \gamma$ (see Lemma 4.17). Since ψ is a δ -approximation, we have

$$\begin{aligned} |p_0, q| &< |\tilde{x}_0, \psi(q)| + \delta = |x_0, \varphi(q)|_{X^{\text{int}}} + \delta \\ &\leq |\varphi(q), \varphi(p_0)|_{X^{\text{int}}} + |\varphi(p_0), x_0|_{X^{\text{int}}} + \delta \\ &\leq |p_0, q|_M + 2\delta. \end{aligned}$$

□

Lemma 6.12. *For any $R > 0$ and $\delta < 1/R$, there exists $\epsilon > 0$ satisfying the following: If M in $\mathcal{M}(n, \kappa, \lambda)$ with $\text{inrad}(M) < \epsilon$ has disconnected boundary ∂M , then every δ -limit $\mathcal{Y}(M, p)$ is double in scale R for every $p \in M$.*

Proof. Suppose that some limit $\mathcal{Y}(M, p) = (Y, X, x)$ is single or mixed in scale R . First note that by Lemmas 6.9 and 6.11, every points q_1, q_2 in $\partial M \cap B^{\tilde{M}}(p, R)$ can be joined by a curve in ∂M . Take a point $p_\alpha \in \partial M$ contained in a component different from that containing p . Let $c : [0, \ell] \rightarrow M$ be a unit speed minimal geodesic in M from p to $p_\alpha \in \partial$. For each i with $1 \leq i \leq [\ell/R]$, take $p_i \in \partial M$ with $|p_i, c(iR)|_M < \epsilon$. By applying Lemmas 6.9, 6.11 and 6.10 to p_i together with a standard monodromy argument, we see that p_α can be joined to p in ∂M , which is a contradiction. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. (1) Suppose that ∂M is disconnected. By Lemma 6.12, every δ -limit $\mathcal{Y}(M, p)$ is double in scale R for every $p \in M$. Take p_α and p_β from distinct components of ∂M . For every $p \in \partial M$, let $c : [0, \ell] \rightarrow M$ be a unit speed curve in M from p_α to p_β through p . For each i with $1 \leq i \leq [\ell/R]$, take $p_i \in \partial M$ with $|p_i, c(iR)|_M < \epsilon$. By applying Lemma 6.10 to each p_i together with a standard monodromy argument, we see that p can be joined to p_α or p_β by a curve in ∂M . Therefore we conclude that the number of boundary components of M is at most two.

(2) Suppose that ∂M has two components. By Lemma 6.10, any δ -limit $\mathcal{M}(M, p) = (Y, X, x)$ is double in scale R for every $p \in \partial M$. Therefore for any $x \in X$, there are distinct $y_1 \neq y_2 \in C_{t_0}$ with $|y_i, x| = t_0$. Take $q_i \in \partial M$, $i = 1, 2$, which are δ -close to y_i in the

Gromov-Hausdorff distance. From Lemma 6.10, q_1 and q_2 must belong to distinct components of $\partial\tilde{M}$, which implies $|q_1, q_2| \geq 2t_0$, and hence $|y_1, y_2| = 2t_0$. Let W be a component of $\partial\tilde{M}$, and consider the distance function d_W from W . The above observation shows that for every $0 < \epsilon_0 < \pi$, d_W is ϵ_0 -regular on a neighborhood of M in \tilde{M} if $\delta = \delta(\epsilon_0, t_0) > 0$ is taken small enough. This means that for any $p \in M$, there exists a point $q \in \partial\tilde{M}$ such that $\tilde{d}Wpq > \pi - \epsilon_0$. This makes it possible to define locally defined gradient-like vector fields for d_W on neighborhoods of the points of M . Then by gluing those local gradient-like vector fields, we get a globally defined gradient-like vector field V on \tilde{M} whose support is contained in a neighborhood of M . It is now straightforward to obtain a diffeomorphism between \tilde{M} and $W \times [0, 1]$ by means of integral curves of V . \square

7. CONVERGENCE WITH NON INRADIUS COLLAPSE

In this section, we consider the situation that M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ converges to a compact length space N while $\text{inrad}(M_i)$ has a positive lower bound. We say that $(\tilde{M}_i, M_i, \partial M_i)$ converges to (Y, X, X_0) if M_i and ∂M_i converge to X and X_0 , $Y \supset X \supset X_0$, under the convergence $\tilde{M}_i \rightarrow Y$. Note that $N = X^{\text{int}}$.

We always assume $m = \dim Y$ as before. From the positive lower bound for $\text{inrad}(M_i)$, we immediately have

$$\dim X = m.$$

It is also easy to see

Lemma 7.1. *X_0 coincides with the topological boundary ∂X of X in Y .*

As before we have a description of Y :

Lemma 7.2. *There is a canonical surjective 1-Lipschitz map $\eta_0 : C_0 \rightarrow X_0$ such that Y is isometric to the length space*

$$X \bigcup_{\eta_0} C_0 \times_{\phi} [0, t_0],$$

where $(x, 0) \in C_0 \times 0$ is identified with $\eta_0(x) \in X_0$ for each $x \in C_0$.

Lemmas 4.1, 4.2, 4.3, Propositions 4.5 and 4.7 still hold if one replaces X by X_0 in this generality. Especially we have the following results. The proofs are similar, and hence omitted.

Lemma 7.3. *For every $x \in X_0$, we have*

- (1) $\#\eta_0^{-1}(x) \leq 2$;
- (2) *for $x \in X_0$, suppose $\#\eta_0^{-1}(x) = 2$, and let γ_{\pm} be the two shortest geodesics from x to C_{t_0} , and let $\xi_{\pm} \in \Sigma_x(Y)$ be the directions of γ_{\pm} respectively. Then $\Sigma_x(Y)$ is isometric to the spherical suspension $\{\xi_{\pm}\} * \Sigma_x(X_0)$.*

Proposition 7.4. *For every $p \in C_0$, any differential $d\eta_0 : T_p(C_0) \rightarrow T_x(X_0)$ satisfies*

$$|d\eta_0(\tilde{v})| = |\tilde{v}|.$$

for every $\tilde{v} \in T_p(C_0)$. In particular, $\eta_0 : C_0 \rightarrow X_0$ preserves the length of Lipschitz curves in C_0 .

For $i = 1, 2$, set

$$X_0^i := \{x \in X_0 \mid \#\eta_0^{-1}(x) = i\}, \quad C_0^i := \eta_0^{-1}(X_0^i).$$

Proposition 7.5. *For every $p \in C_0^2$, we have*

- (1) *any differential $d\eta_p$ provides an isometry $d\eta_p : T_p(C) \rightarrow T_x^+(Y)$ which preserves the half suspension structures of both $\Sigma_p(C) = \{\xi_+\} * \Sigma_p(C_0)$ and $\Sigma_x^+(Y) := \{\xi_+\} * \Sigma_x(X_0)$, where $T_x^+(Y) = T_x(X_0) \times \mathbb{R}_+$;*
- (2) *$p \in C_0^{reg}$ if and only if $x \in X_0^{reg}$. In this case, $(d\eta_0)_p : T_p(C_0) \rightarrow T_x(X_0)$ is a linear isometry, where $X_0^{reg} := X_0 \cap Y^{reg}$.*

As an application of Lemma 7.3, we have the following result, which gives a sufficient condition for inradius collapse.

Proposition 7.6. *Let M_i in $\mathcal{M}(n, \kappa, \lambda, d)$ converge to a compact length space N with respect to the Gromov-Hausdorff distance, and suppose that N is a closed topological manifold or a closed Alexandrov space. Then $\text{inrad}(M_i)$ converges to zero.*

Proof. We assume that N is a closed Alexandrov space with curvature bounded below. The case when N is a closed topological manifold is similar. Suppose that Proposition 7.6 does not hold. Let $r_i := \text{inrad}(M_i)$, and take a point $p_i \in M_i$ and $q_i \in \partial M_i$ such that $|p_i, q_i| = r_i$. Passing to a convergence, we may assume that $(B(p_i, r_i), q_i)$ converges to a metric ball $(B(x_0, r), y_0)$ in X under the convergence $\tilde{M}_i \rightarrow Y$, where $r > 0$.

Take a minimal geodesics γ and γ^+ from y_0 to x_0 and C_{t_0} respectively. Note that

$$(7.15) \quad \angle(\gamma, \gamma^+) = \pi$$

We claim that $\#\eta_0^{-1}(y) = 1$ for every point of X_0 near y_0 . Otherwise we have a sequence $y_i \in X_0$ converging to y_0 with $\#\eta_0^{-1}(y_i) = 2$. Take two minimal geodesics γ_i^\pm from y_i to C_{t_0} . For every $\delta > 0$ take $s_0 > 0$ such that $\tilde{\angle}\gamma(s_0)y_0\gamma_+(s_0) > \pi - \delta$. Since γ_i^\pm converges to γ^\pm , we obtain

$$\angle\gamma(s_0)y_i\gamma_i^\pm(s_0) \geq \tilde{\angle}\gamma(s_0)y_i\gamma_i^\pm(s_0) > \pi - 2\delta,$$

for large enough i . This implies that $\angle(\gamma_i^+, \gamma_i^-) < 2\delta$ contradicting to Lemma 7.3.

Let $p_0 \in C_0$ with $\eta_0(p_0) = y_0$, and set $a := \gamma^+(t_0)$. From the previous consideration, it is possible to take neighborhoods U_0 of p_0 in C_0 , V_0 of y_0 in X_0 respectively in such a way that $\eta_0 : U_0 \rightarrow V_0$

is a homeomorphism. From (7.15), we may assume that the distance function d_a from a is regular on U_0 . Choose a neighborhood $U_1 \subset U_0$ homeomorphic to \mathbb{R}^{m-1} . Perelman's fibration theorem now implies that a small neighborhood of any point $y \in \eta_0(U_1)$ in Y is homeomorphic to \mathbb{R}^m . On the other hand, since X is bi-Lipschitz homeomorphic to a closed Alexandrov space and since $\dim \eta_0(U_1) = m - 1$, one can take a point $y \in \eta_0(U_1)$ having a neighborhood in X homeomorphic to \mathbb{R}^m . Since y is a boundary point of X in Y , this contradicts to the domain invariance theorem in \mathbb{R}^m . This completes the proof. \square

We now study local geometric properties of $X_0 = \partial X$

Let ∂X_0^i and ∂C_0^i denote the topological boundaries of X_0^i and C_0^i in X_0 and C_0 respectively. Recall that $Y = X \bigcup_{\eta_0} C_0 \times_\phi [0, t_0]$. Since $\#\eta_0^{-1}(x) \leq 2$ for every $x \in X_0$, we have an involutive map $f : C_0 \rightarrow C_0$ as before. However, f is not continuous in general.

Lemma 7.7. $f : C_0 \rightarrow C_0$ is continuous on $C_0^1 \cup (C_0 \setminus \partial C_0^i)$.

Proof. The proof is similar to that of Lemma 4.8, and hence omitted. \square

Remark 7.8. Remark that the final part of the proof of Lemma 4.8, which is the case when $p_i \in C_0^1$ converges to $p \in C_0^2$, does not work in the present situation. Lemma 7.7 shows that this is the only case where the continuity of f does not hold.

We denote by $\tilde{\mathcal{S}}$ the topological boundary of C_0^1 and C_0^2 in C_0 , and by \mathcal{S} the topological boundary of X_0^1 and X_0^2 in X_0 :

$$\tilde{\mathcal{S}} := \partial C_0^1 = \partial C_0^2, \quad \mathcal{S} := \partial X_0^1 = \partial X_0^2.$$

In view of $N = X^{\text{int}}$, we set

$$N_0 := X_0, \quad N_0^i := X_0^i,$$

as in Section 1. We call \mathcal{S} the *topological boundary singular set* of X or N . For each $i = 1, 2$, a point of

$$\mathcal{S}^i := \mathcal{S} \cap X_0^i$$

is called a *topological boundary singular point of type i* .

Example 7.9. For $0 < \epsilon \ll \delta < 1$, let $f_\epsilon : (-3, 3) \rightarrow (0, \infty)$ be a smooth function satisfying

- (1) $f_\epsilon(-x) = f_\epsilon(x)$, $\lim_{x \rightarrow \pm 3} f_\epsilon(x) = 0$,
- (2) $f_\epsilon(x)$ does not depend on ϵ on $|x| \geq 1$,
- (3) $f_\epsilon(x) = \epsilon$ on $[-\delta, \delta]$,
- (4) $f'_\epsilon > 0$ on $(\delta, 2)$, and $f'_\epsilon < 0$ on $(2, 3)$
- (5) $|f''_\epsilon|$ is uniformly bounded on $[\delta, 1]$,
- (6) the domain N_ϵ on xy -plane bounded by the graphs $y = \pm f_\epsilon(x)$ is smooth.

Note that ∂N_ϵ has a uniformly bounded absolute geodesic curvature. therefore $M_\epsilon := N_\epsilon \times S_\epsilon^1$ belongs to $\mathcal{M}(3, 0, \lambda, d)$ for some λ and d , and it converges on N , which is the limit of N_ϵ for the Hausdorff distance. In this case, N_0 coincides with the topological boundary of N in \mathbb{R}^2 , $N_0^2 = [-\delta, \delta] \times 0$ and $\mathcal{S} = \mathcal{S}^2$ consists of the two points $\{(\pm\delta, 0)\}$.

In a way similar to Corollary 4.9, Lemmas 4.13, 4.14 and 4.15, we have the following

Lemma 7.10. *The maps η_0 and f have the following properties:*

- (1) $\eta_0|_{\text{int}C_0^2} : \text{int}C_0^2 \rightarrow \text{int}X_0^2$ is a locally isometric double covering space;
- (2) $\eta_0|_{\text{int}C_0^1} : \text{int}C_0^1 \rightarrow \text{int}X_0^1$ is an isometry;
- (3) $f : (C_0)^{\text{int}} \setminus \tilde{\mathcal{S}} \rightarrow (C_0)^{\text{int}} \setminus \tilde{\mathcal{S}}$ is an isometry.

Lemma 7.11. $X_0^2 \setminus \partial X_0^2$ is open in X . In particular,

- (1) for every $x \in X_0^2$, there is an $\epsilon > 0$ such that $\dim_H X \cap B(x, \epsilon) = m - 1$;
- (2) \mathcal{S} is empty if and only if $X_0 = X_0^1$.

Proof. In view of Lemma 7.10, the proof is straightforward, and hence omitted. \square

From the fact that C_0 is an Alexandrov space, we immediately have

Corollary 7.12. $X_0 \setminus \mathcal{S}$ equipped with the interior metric is an Alexandrov space with curvature $\geq c(\kappa, \lambda)$, where $c(\kappa, \lambda)$ is a constant depending only on κ and λ .

Let M_i converges to $N = X^{\text{int}}$, and let $\partial_{\text{top}} N := \partial_{\text{top}} X^{\text{int}}$. In the case of non inradius collapse, N does not satisfy the condition of Alexandrov spacs at $\partial_{\text{top}} N$ in general. Nevertheless, we can define the regularity of N as follows. Note that $N \setminus \partial_{\text{top}} N$ locally satisfies the condition of Alexandrov space with curvature $\geq \kappa$.

Theorem 7.13. *Let M_i converges to N while $\text{inrad}(M_i)$ has a positive lower bound. Suppose that*

- (1) $N \setminus \partial_{\text{top}} N$ is almost regular or almost regular with almost regular boundary;
- (2) $\lim_{GH} (\partial M_i)^{\text{int}}$ is almost regular;
- (3) \mathcal{S} consists of only topological boundary singular points of type 2.

Then there exists a gluing of some closed domains $(M_i)_{\text{fib}}$ and $(M_i)_{\text{cap}}$ of M_i along their boundaries:

$$M_i = (M_i)_{\text{fib}} \cup_{L_i} (M_i)_{\text{cap}},$$

together with fiber bundles

$$\begin{aligned} F_i &\longrightarrow (M_i)_{\text{fib}} \longrightarrow N^*, \quad \text{Cap}_i \longrightarrow (M_i)_{\text{cap}} \longrightarrow \partial_{\text{alex}} N, \\ F_i &\longrightarrow L_i \longrightarrow \partial_{\text{alex}} N, \end{aligned}$$

where the fibers F_i are almost nonnegatively curved closed manifolds, N^* is an almost regular Alexandrov space with almost regular boundary having the same Lipschitz homotopy type as N , Cap_i are manifolds with boundary homeomorphic to F_i , and those fiber structures are compatible to each other.

In case $\partial_{\text{alex}} N$ is empty, we have a fiber bundle

$$F_i \longrightarrow (M_i) \longrightarrow N^*.$$

It should be noted that in Theorem 7.13 N has singular points if \mathcal{S} is non-empty.

Proof. From the assumption, C_0 is an almost regular Alexandrov space with curvature bounded below. It follows from Lemma 7.3 and Proposition 7.5 that every $x \in \mathcal{S}$ is an almost regular point of Y when $N = X^{\text{int}}$ is considered as a subset of Y . In view of Lemma 7.10, we conclude that the Alexandrov space Y is almost regular with almost regular boundary in the sense of Section 2.2. Consider the double $D(M_i)$ of M_i and the partial double

$$D_0(Y) = Y \amalg_{C_{t_0}} Y,$$

where the two copies of Y are glued along C_{t_0} , which is an Alexandrov space. Note that both $D(M_i)$ and $D_0(Y)$ have canonical \mathbb{Z}_2 -actions defined by reflections. Applying the fibration-capping theorem in [Ym] to the convergence $(D(M_i), \mathbb{Z}_2) \rightarrow (D_0(Y), \mathbb{Z}_2)$, we have a \mathbb{Z}_2 -invariant gluing

$$D(M_i) = D(M_i)_1 \cup_{L_i} D(M_i)_2 \cup_{L_i} D(M_i)_3$$

by some closed domains $D(M_i)_k$, $1 \leq k \leq 3$, of together with compatible \mathbb{Z}_2 -equivariant fibrations

$$F_i \longrightarrow D(M_i)_2 \longrightarrow D_0(Y), \quad Cap_i \longrightarrow D(M_i)_k \longrightarrow \partial_{\text{alex}} Y = \partial_{\text{alex}} N,$$

for $k = 1, 3$. Obviously these fibrations descends to fibrations

$$F_i \longrightarrow (M_i)_{\text{fib}} \longrightarrow Y, \quad Cap_i \longrightarrow (M_i)_{\text{cap}} \longrightarrow \partial_{\text{alex}} N, .$$

Since Y has the same Lipschitz homotopy type as N ([32], this completes the proof. \square

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